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Linear Algebra

Vector Space and Its Basic Properties

Definitions

Internal Composition: Let A be a set. Then the mapping $f: A \times A \rightarrow A$ is said to be internal composition and it also called vector addition.

Example: Consider R , the set of all real numbers Let $f: R \times R \rightarrow R$ be defined as: $f((a, b)) = ab \forall (a, b) \in R \times R, a, b \in R$.

Then f is an internal composition in R .

External Composition: Let V and F be any two non-empty sets, then the mapping $f: V \times F \rightarrow V$ is said to be an external composition in V over F . Also called scalar multiplication

Vector Spaces: Let $(F, +, \cdot)$ be the given field then the elements of F are called scalars. And let V , be a non-empty set, then the elements of V are called vectors.

V is said to be a vector space over the field F then the elements of F are called scalars where as if the following postulates are satisfied under vector addition '+' and scalar multiplication '·'.

1. $(V, +)$ is abelian group
2. $\forall a \in F$ and $\forall x \in V, ax \in V$. Closed under scalar multiplication
3. $a(a + y) = ax + ay \forall a \in F$ and $\forall x, y \in V$
4. $(a + b)x = ax + bx \forall a, b \in F$ and $\forall x \in V$
5. $(ab)x = a(bx), \forall a, b \in F, \&x \in V$
6. $x = x), \forall x \in V$

Note:

- 1 is the unity element in the field F
- O is the zero element of V and O is the zero element of F .
- The vector space of V over the field F is denoted as $V(F)$.
- $V(F)$ is a real vector space if F is the field R of real numbers.
- $V(F)$ is a rational vector space if F is the field Q of rational numbers.
- $V(F)$ is a complex vector space if F is the field C of complex numbers.

Properties On Vector Space

1. If x, y , and z are vectors in a vector space v
Such that $x + z = y + z$,
Then $x = y$
2. In any vector space v , the following statements are true
 - a. $Ox = O$ FOR EACH $x \in V$
 - b. $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in V$
 - c. $aO = O$ for each $a \in F$.

Subspace: a subset w of a vector space v over field f is called a subspace of v if w is a vector space over f with the operations of addition and scalar multiplication defined on v .

Improper and Proper Subspaces: if v is any vector space then v itself is a subspace of v . the subset consisting of the zero vector alone i.e. $\{0\}$ is a subspace of v always, called the zero subspace of v . these two subspaces are called improper subspaces. the subspaces other than $\{0\}$ and v are called proper subspaces.

Tests for showing subspaces

Two-step test: a subset w of a vector space $v(f)$

is a subspace of v if

1. W is non-empty

$$\forall x, y \in W \Rightarrow x - y \in W$$

$$\forall a \in F, x \in W \Rightarrow ax \in W$$

One-step test: a non-empty subset w of a vector space $V(F)$ is a subspace of $V(F)$ if $ax + \beta y \in W, \forall a, \beta \in F$ and $\forall x, y \in W$

Propositions on Subspaces

- let v be a vector space over f . a nonempty subset w of v is a subspace of v if and only if $\alpha u + \beta v \in W$ for all $\alpha, \beta \in F$, whenever $u, v \in W$
- any intersection of subspaces of a vector space v is a subspace of v
- let W_1 and W_2 be subspaces of a vector space v then $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$
- if S_1 and S_2 are nonempty subsets of a vector space v , then the sum of S_1 and S_2 , denoted by $S_1 + S_2$ is defined as $\{x + y : x \in S_1 \text{ and } y \in S_2\}$
- let W_1 and W_2 be subspaces of a vector space v then the sum $W_1 + W_2$ is a subspace of v that contains both W_1 and W_2
- any subspace of v that contains both W_1 and W_2 must also contain $W_1 + W_2$ i.e., $W_1 + W_2$ is the smallest subspace contains both W_1 and W_2

Direct Sum: if W_1 & W_2 be two subspaces of a vector space $v(f)$ then the internal direct sum of W_1 & W_2 is denoted by $W_1 \oplus W_2$ and each element of $W_1 \oplus W_2$ can be expressed as a sum of unique elements of W_1 & W_2 i.e. if every element of V can be expressed as a sum of unique element of W_1 & W_2 Then $V = W_1 \oplus W_2$

Note: If W_1 & W_2 be two subspaces of a vector space $V(F)$ such that $V = W_1 \oplus W_2$ then W_1 & W_2 are called complement to each other.

Theorem: A vector space V is called the Direct sum of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. V is the direct sum of W_1 and W_2 then we writing $V = W_1 \oplus W_2$

Example: Consider the vector space $V = R^2$ over real field R and $W_1 = \{(x, 0) \mid x \in R\}$, $W_2 = \{(0, y) \mid y \in R\}$, $W_3 = \{(x, x) \mid x \in R\}$ are three subspace of V then $V = W_1 \oplus W_2$, $V = W_2 \oplus W_3$ and also $V = W_1 \oplus W_3$

Note: Complement Subspace may or may not be unique.

Orthogonal Complement Subspace: If W_1 & W_2 be two subspaces of a vector space $V(F)$ such that $V = W_1 \oplus W_2$ then W_1 are called Orthogonal complement of W_2 subspace if each vector of W_1 is orthogonal to each vector of W_2 .

Example:

- Let $V = R^2$ and let $W = \{(a, 0) : a \in R\}$, $W_1 = \{(0, b) : b \in R\}$ and $W_2 = \{(c, c) : c \in R\}$ then $V = W \oplus W_1$ and $V = W \oplus W_2$.

Note: Subspace W and W_1 are orthogonal complement subspace. W_1 and W_2 are complement subspace but not orthogonal.

- Let $V = R^2(R)$ and let $W_1 = \{(x, y, 0) : x, y \in R\}$ and $W_2 = \{(x, 0, z) : x, z \in R\}$ be two subspaces of V . then $V = W_1 + W_2$. But V is not the direct sum of W_1 and W_2 .
- If V is a vector space of all real valued function over R then it is a direct sum of its subspace of even and odd functions.
- A vector space of all $n \times n$ matrices is direct sum of subspace of symmetric and skew symmetric matrices.
- A vector space of all $n \times n$ matrices with complex entries is direct sum of subspace of Hermitian and skew Hermitian matrices
- If M, N_1 and N_2 are three subspaces of a vector space V such that
- $M \oplus N_1 = M \oplus N_2 = V, \neq N_1 \neq N_2$

Example: Let $V = R^3$ and $M = (a, b, 0) : a, b \in R, N_1 = (0, b, b) : b \in R, N_2 = (0, 0, c) : c \in R$

- $V = R[x]$ Polynomial space over R and $W_1 = \{p(x) \in V \mid p(0) = 0, p'(0) = 0, \dots, p^{(k)}(0) = 0\}$ where $p^{(k)}(x)$ denotes the k^{th} derivative of $p(x)$, $W_2 = \{p(x) \in V \mid \deg(p(x)) \leq k\}$ then $V = R[x]$ is direct sum of W_1 & W_2 .
- $V = R$ be a vector space over field R then

$$w_1 = \{(a_1, a_2, \dots, a_n) \mid a_i = 0, \text{ if } i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} \text{ and } w_2 \\ = \{(a_1, a_2, \dots, a_n) \mid a_i = 0, \text{ if } i = \lfloor \frac{n+1}{2} \rfloor, \dots, n\} \text{ then}$$

V is direct sum of W_1 and W_2 iff n is odd

Cosets of Subspace: Let W be a subspace of a vector space V over a field F . For any $v \in V$ the set $v + W = \{v + w \mid w \in W\}$ is called the coset of W containing v . Denote this coset by $v + W$ rather than $v + W$.

Quotient Space: Let W be a subspace of a vector space V over a field F and $S = \{v + W : v \in V\}$ of all cosets of W then set S is a vector space over a field F with the operations internal composition $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$ and external composition is $a(v + W) = av + W$. This vector space is called the quotient space of V modulo W and is denoted by $\frac{V}{W}$

Propositions:

- Let W be a subspace of a finite dimensional vector space $V(F)$. Then

$$\dim\left(\frac{V}{W}\right) = \dim(V) - \dim(W)$$

- Let W be a finite dimensional subspace of an infinite dimensional vector space $V(F)$. Then $\dim\left(\frac{V}{W}\right) =$ infinite dimension
- Let W be an infinite dimensional subspace of an infinite dimensional vector space $V(F)$. Then $\dim\left(\frac{V}{W}\right)$ can be infinite dimension or finite dimension

Examples:

- Let V be the vector space of all 2×2 matrices over C . And $w_1 = \{[x \ y \ z \ 0] : x, y, z \in C\}, w_2 = \{[x \ 0 \ 0 \ y] : x, y \in C\}$, then $\dim\left(\frac{w_1+w_2}{w_1 \cap w_2}\right) = \dim\left(\frac{w_1}{w_1 \cap w_2}\right) = 2$
- Let V the vector space of all 2×2 matrices over C . $w = \{[x \ -x \ y \ z] : x, y, z \in C\}, w_2 = \{[a \ b \ -a \ c] : a, b, c \in C\}$ then $\dim\left(\frac{w_1+w_2}{w_1 \cap w_2}\right) = \dim\left(\frac{w_1}{w_1 \cap w_2}\right) = 1$

Propositions Direct Sum And Quotient Space

1. Let W_1 and W_2 be subspaces of a vector space V . Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$
2. $v + W$ is a subspace of V if and only if $v \in W$.
3. $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.
4. Let W be a subspace of a vector space V over a field F . $S = \{v + W : v \in V\}$ of all cosets of W then $a(v + W) = av + W$

Spanning Of Vector Spaces: We're interested in spanning down what it means for a vector space to have a basis, and that's described in terms of the concept of linear combination. Span and independence are two more related concepts.

Linear Combination: a linear combination is an expression constructed from a set of terms by multiplying each term by a constant and adding the results (e.g. a linear combination of x and y would be any expression of the form $ax + by$, where a and b are constants). The concept of linear combinations is central to linear algebra and related fields of mathematics.

Let $V(F)$, be a vector space over field F and $S \subseteq V$ be a subset of V may be finite or infinite then for any vectors $x_1, x_2, \dots, x_n \in S$ and scalars $a_1, a_2, \dots, a_n \in F$ implies $x = \sum_{i=1}^n a_i x_i \in V$ is defined as linear combination of vectors $x_1, x_2, \dots, x_n, x \in S$ or x is in linear combination of vectors x_1, x_2, \dots, x_n

EXAMPLE: Let $S \subset V(F)$ where $S = \{x_1, x_2\}$ and F is finite field say $F = \{a_1, a_2, a_3\}$, then $a_1x_1, a_2x_1, a_3x_1, a_1x_2, a_2x_2$ etc. are different linear combination of the elements of S .

Linear Span Or Spanning Set Or Generating Set: The linear span of a non-empty subset S of Vector space $V(F)$ is defined as the set consisting of all linear combinations of the elements of S and denoted by $L(S)$. or $\langle S \rangle$ or $Span(S)$

OR

"set of all possible linear combinations of the elements of S is called the linear span of S "

In fact $L(S)$ is the smallest subspace of V containing S in other words we can say that $L(S)$ is the intersection of all the subspaces containing S .

Theorem: Let $V(F)$ be a vector space and $S \subseteq V$ then $L(S)$ is subspace of V called subspace spanned by S or generated by S .

Note: As ϕ is the subset of all the vector subspace of V and if we take intersection of all the subspaces of V the resulting subspace is $\{0\}$ hence we can say $L(\phi) = \{0\}$.

Example: Let V be the vector space of all polynomial function over F . Let S be the subset of V consisting of the polynomial functions f_0, f_1, f_2, \dots defined by $f_n(x) = x^n, n = 0, 1, 2, \dots$ then V is the subspace spanned by the set S .

Linear Dependence And Independence

Linear Dependence: Let V be a vector space over F . A subset S of V is said to be dependent if there exists distinct vectors a_1, a_2, \dots, a_n in S and scalar c_1, c_2, \dots, c_n in F , not all of which are 0, such that $c_1x_1 + c_2x_2 + \dots + c_nx_n = 0$

If the set S contains finitely many vectors x_1, x_2, \dots, x_n , then sometimes we can say that the vectors x_1, x_2, \dots, x_n are dependent.

In other words if x_1, x_2, \dots, x_n are linearly dependent. Then $\exists c_1, c_2, \dots, c_n$ not all zero s.t. $c_1x_1 + c_2x_2 + \dots + c_nx_n = 0$ Without loss of generality we can assume $c_1 \neq 0$ then $x_1 = (-c_1^{-1}c_2)x_2 + (-c_1^{-1}c_3)x_3 + \dots + (-c_1^{-1}c_n)x_n$, hence we can say x_1 is linear combinations of the others.

Linear Independent: Any set containing the vectors x_1, x_2, \dots, x_n defined over a field F is said to be linearly independent if it is not linear dependent i.e. if $c_1x_1 + c_2x_2 + \dots + c_nx_n = 0$

Implies $c_i = 0$ for all i .

Example: In R^3 the vectors $x_1 = (3,0,-3), x_2 = (-1,1,2), x_3 = (4,2,-2), x_4 = (2,1,1)$ are linearly dependent since $2x_1 + 2x_2 - x_3 + 0x_4 = 0$

Example: Let V be the vector space of 2×2 matrices over R . Determine whether the matrices $A, B, C \in V$ are dependent where:

$$A = [1 \ 1 \ 1 \ 1], C = [1 \ 0 \ 0 \ 1], C = [1 \ 1 \ 0 \ 0]$$

Example: Let V be the vector space of functions from R into R . Show that $f, g, h \in V$ are independent where: (i) $f(t) = e^{2t}, g(t) = t^2, h(t) = t$; (ii) $f(t) = \sin t, g(t) = \cos t, h(t) = t$.

Vector Space Spanned By Subset: $V(F)$, be a Vector Space and $S \subseteq V$ then V is called spanned by S if $L(S) = V$

Finitely Generated Vector Space (FGVS) And Non-Finitely Generated Vector Space (NFGVS): A Vector Space $V(F)$. is called finitely generated vector space if there exists finite $S \subseteq V$ such that $L(S) = V$ otherwise non-finitely Generated Vector Space (NFGVS)

Example: $V = R^3$ is vector space over R is finitely generated vector space. As $S = \{(1,0,0) (0,1,0) (0,0,1)\} \subset R^3$ spanned $V = R^3$

1. $R(x)$ is non-finitely generated vector space (nfgvs)
2. $V = R[x]$ is polynomial space over field R is non-finitely generated vector space (nfgvs)
3. $V = \{f: R \rightarrow R \mid f \text{ is function}\}$ then V is vector space over field R . Under addition of functions and scalar multiplication to the functions. $V(R)$ is non-finitely generated vector space (nfgvs)

Basis and dimension: let V be a vector space, a basis for V is a linearly independent set of vectors in V which spans the space V . The space V is finite dimensional if it has a finite basis i.e. the basis has finite number of elements i.e., let $V \subset V(F)$, then S is a basis if

1. S is linearly independent
2. $L(S) = V(F)$

Dimension Of Vector Space: Dimension of a finite dimensional vector space is the number of vectors in any basis set of the vector space and it is denoted by $\dim(V)$

Example:

1. $C(C)$ is a vector space of dimension one as $S = \{1\}$ is Linearly independent set which spans $C(C)$
2. $C(R)$ is a vector space of dimension two as $S = \{1, t\}$ is linearly independent set which spans $C(R)$
3. $V = R[x]$ is polynomial space over field R is infinite dimension vector space.
4. $V = \{f: R \rightarrow R \mid f \text{ is function}\}$ then V is vector space over field R . Under addition of functions and scalar multiplication to the functions. $V(R)$ is infinite dimension vector space.

Ordered Basis: If V is a finite dimensional vector space, then an ordered basis for V is a finite sequence of vectors which is linearly independent and spans V .

i.e. let $S = \{x_1, x_2, x_3, \dots, x_n\}$ be a basis of $V(F)$ then it is called an ordered basis if the position of vectors in S is fixed i.e. the set S is ordered, hence if $k = \{x_1, x_2, x_3\}$ is an ordered basis then the basis $\{x_2, x_1, x_3\}$ is considered as different ordered basis.

Properties On Linearly Independent And Dependent And Basis

1. Every subset of linearly independent set is linearly independent. That is let $v(f)$ be a vector space, and let $S_1 \subseteq S_2 \subseteq v$. If S_2 is linearly independent, then S_1 is linearly independent

2. The empty set is linearly independent, hence every linearly dependent sets must be nonempty.
3. A set consisting of a single nonzero vector is linearly independent. For if $\{u\}$ is linearly dependent, then $au = 0$ for some nonzero scalar a
4. A set linearly independent if and only if the only representations of 0 as linear combinations of its vectors are trivial representations
5. A set linearly dependent if and only if the on representations of 0 as linear combinations of it. vectors are non-trivial representations
6. Any set containing 0 is linearly dependent
7. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S is linearly dependent, then S_2 is linearly dependent.
8. Let S be linearly independent subset of a vector space and let v be a vector in V that is not in S . Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$
9. Let u and v be distinct vectors in vector space V . then $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other
10. Let V be a vector space over a field of characteristic not equal to two.
 - a. Let u and v be distinct vectors in V , then $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.
 - b. Let u, v and w be distinct vectors in V . Then $\{u, v, w\}$ is linearly independent if and only if $\{u + v, v + w, u + w\}$ is linearly independent
11. Let $S = \{u_1, u_2, \dots, u_n\}$ be a finite set of vectors. Then S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \dots, u_k\})$ for some $k(1 \leq k < n)$
12. A set S of vectors is linearly independent if and only if each finite subset of S is linearly independent
13. S be a set of nonzero polynomials in $P(F)$ such that the same degree. then S is linearly independent
14. if $\{A_1, A_2, \dots, A_k\}$ is a linearly independent subset of $M_{m \times n}(F)$, then $\{A'_1, A'_2, \dots, A'_k\}$ is also linearly independent
15. Let $f, g \in F(R, R)$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Then that f and g are linearly independent in $F(R, R)$.
16. Let V be a vector space and $\beta = \{u_1, u_2, \dots, u_n\}$ be a subset of V . Then β is a basis for V . If and only if each $v \in V$ can be uniquely expressed as linear combination of vectors of β , that is, can be expressed in the form $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$
17. If a vector space V is generated by a finite set S , then some subset of S is a basis for V , i.e. V has a finite basis
18. A vector space V over a field K is said to be finite dimensional if there are finite number of elements x_1, \dots, x_n in V such that $V = \langle x_1, \dots, x_n \rangle$. If no such
19. If vector space of continuously differentiable functions is taken then any subset of this vector space is linearly independent if Wronskian of the functions is none zero.
20. Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors
21. Let V be a vector space with dimension n
 - a. Any finite generating set for V contains at least n vectors, and generating set V that contains exactly n vectors is a basis for V .
 - b. Any linearly independent subset of V that contains exactly n vectors is a basis for V .
 - c. Every linearly independent subset of V can be extended to a basis for V .
22. Let W be a subspace of finite-dimensional vector space V . Then W is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

23. If W is a subspace of finite-dimensional vector space V , then any basis for W can be extended to basis for V
24. if W_1 and W_2 are finite-dimensional subspaces of a vector space V , then
- The subspace $W_1 + W_2$ is finite-dimensional, and
 - $W_1 + W_2$ is smallest subspace of V containing $W_1 \cup W_2$ i.e. $L(W_1 \cup W_2) = W_1 + W_2$
 - $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$
25. Let W_1 and W_2 be subspaces of vector space V having dimensions m and n , respectively, where $m \geq n$ Then
- $\dim(W_1 \cap W_2) \leq n$
 - $\dim(W_1 + W_2) \leq m + n$
26. Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$
- If β_1 and β_2 are bases for W_1 and W_2 , respectively, then $\beta_1 \cap \beta_2 = \phi$ and $\beta_1 \cup \beta_2$ is a basis for V
 - Let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2 , respectively, or a vector space V if $\beta_1 \cup \beta_2$ is a basis for V , then $V = W_1 \oplus W_2$
27. Let W be a subspace of a (not necessarily finite-dimensional) vector space V then any basis for W is a subset of basis for V
28. let V be a finite dimensional vector space over field K , and let X and Y be finite subsets of V . If Y is linearly independent and $V = \langle X \rangle$, then $|Y| \leq |X|$
29. Let V be finite dimensional vector space over field K . If X is a finite dimensional subspace of V is finite
30. Let V be finite dimensional vector space over K . The the following statements are equivalent for a subset β of V
- β is a basis
 - β is a minimal generating set, that is, no subset of β can generate V
 - β is a maximal linearly independent set
31. Let V be a vector space of dimension n . Then every linearly independent subset of V with n elements is a basis of V
32. $\frac{V}{W} \subset P(V)$ but never contained in V .
33. If x_i 's are linearly independent in V then, it is not necessary that $(W + x_i)$'s are linearly independent
34. If x_i 's are linearly dependent in V then $(W + x_i)$'s are also linearly dependent in $\frac{V}{W}$.
35. For every FDVS V there exists subspaces W_1 and W_2 s.t. $V = W_1 \oplus W_2$.
36. $\dim(W_1 \cap W_2) \leq \min[\dim(W_1), \dim(W_2)]$
37. $\dim(W_1 \cap W_2) \leq \max[0, \dim(W_1) + \dim(W_2) - \dim(V)]$
38. $\max[\dim(W_1), \dim(W_2)] \leq \dim(W_1 + W_2) \leq \min[\dim(W_1) + \dim(W_2), \dim(V)]$
39. If $P_n(x, y)$ denotes vector space of polynomial with degree less than or equal to n in indeterminate x and y then following is true
- $\dim(P_n(x, y)) = \frac{(n+1)(n+2)}{2!}$
 - $\dim(P_n(x, y, z)) = \frac{(n+1)(n+2)(n+3)}{3!}$ and can be generalized.

Counting of Ordered Linear Independent Set and Subspace

If $\dim V(F) = n$ and $\text{Card}(F) = p^k$ where n is dimension of V over F and F is finite, then

- $|V(F)| = p^{nk}$ i.e. $|V(F)| = |F|^n$
- Number of unordered basis = $\frac{\text{Number of ordered basis}}{n!}$

3. The number of linearly independent subsets of cardinality l are $= \frac{(p^{nk-1})(p^{nk-p}) \dots (p^{nk-p^{(n-1)k}})}{k!}$
4. The number of ordered linearly independent subsets of cardinality k are. The number of basis of $V = (p^n - 1)(p^n - p) \dots (p^n - p^{k-1})$
5. The number of basis of $V = \frac{(p^n-1)(p^n-p) \dots (p^n-p^{n-1})}{n!}$
6. The number of ordered basis of $V = (p^n - 1)(p^n - p) \dots (p^n - p^{n-1})$

Note: Number of ordered basis is actually the order of the group $GL(n, F)$

1. The number of subspaces of dimension k of $V =$

$$\frac{(p^n - 1)(p^n - p) \dots (p^n - p^{k-1})}{(p^k - 1)(p^k - p) \dots (p^k - p^{k-1})}$$

2. Number of subspaces of $V(F)$, defined as above with dimension r

$$\frac{\prod_{i=0}^{r-1} (p^{nk} - p^{ki})}{\prod_{i=0}^{r-1} (p^{kr} - p^{ki})}$$

3. Let $|K| = p^r$ and $|F| = p^n$ and $K < F$ then r/n .

If V is n -dimensional vector space over field F , where $|F| = p$, then

Examples on Vector Spaces and Their Dimension

1. Let $V(F)$ denotes the V is a Vector space over field F then which of the following is a Vector space under usual additions of vectors and scalar multiplication?
2. Every field is Vector space over its subfield is a vector space, i.e. if F' is subfield of F then $F(F')$ is a Vector space

| | Structure | Vector Space | Dimension |
|-----|----------------------------|--------------|--------------------|
| 1. | $C(C)$ | Yes | 1 |
| 2. | $C(R)$ | Yes | 1 |
| 3. | $C(Q)$ | Yes | Infinite dimension |
| 4. | $C(Q(\sqrt{2}))$ | Yes | Infinite dimension |
| 5. | $R(C)$ | NO | - |
| 6. | $R(R)$ | Yes | 1 |
| 7. | $R(Q)$ | Yes | Infinite dimension |
| 8. | $R(Q(\sqrt{2}))$ | Yes | Infinite dimension |
| 9. | $R(Q^c)$ | No | - |
| 10. | $Q(C)$ | No | - |
| 11. | $Q(R)$ | No | - |
| 12. | $Q(Q)$ | Yes | 1 |
| 13. | $Q(Q(\sqrt{2}))$ | No | - |
| 14. | $Q(Q^c)$ | No | - |
| 15. | $Q(\sqrt{2})(C)$ | No | - |
| 16. | $Q(\sqrt{2})(R)$ | No | - |
| 17. | $Q(\sqrt{2})(Q)$ | Yes | 2 |
| 18. | $Q(\sqrt{2})(Q(\sqrt{2}))$ | Yes | 1 |

3. Let F' is subfield of F and let $F^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in F\}$ set of n -tuple's of field then $F^n(F')$ is Vector Space

| C^n | | | |
|-------|---------------|--------------|--------------------|
| | Over | Vector Space | Dimension |
| 1. | C | Yes | n |
| 2. | R | Yes | $2n$ |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

| R^n | | | |
|-------|---------------|--------------|--------------------|
| | Over | Vector Space | Dimension |
| 1. | C | No | - |
| 2. | R | Yes | n |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

| Q^n | | | |
|-------|---------------|--------------|-----------|
| | Over | Vector space | Dimension |
| 1. | C | No | - |
| 2. | R | No | - |
| 3. | Q | Yes | n |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | No | - |

| $Q(\sqrt{2})^n$ | | | |
|-----------------|---------------|--------------|-----------|
| | Over | Vector Space | Dimension |
| 1. | C | No | - |
| 2. | R | No | - |
| 3. | Q | Yes | $2n$ |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | In |

4. Let F' is subfield of F and let $V = F^{n \times m} = \{A = [a_{ij}] \mid a_{ij} \in F\}$ set of $n \times m$ matrices over field F , $V(F')$ is vector space over field F' under matrix addition and scalar multiplication $C^{n \times m}$

| $C^{n \times m}$ | | | |
|------------------|---------------|--------------|--------------------|
| | Over | Vector space | Dimension |
| 1. | C | Yes | $m \cdot n$ |
| 2. | R | Yes | $2m \cdot n$ |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

| $R^{n \times m}$ | | | |
|------------------|---------------|--------------|--------------------|
| Over | | Vector space | Dimension |
| 1. | C | No | - |
| 2. | R | Yes | m.n |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

| $Q^{n \times m}$ | | | |
|------------------|---------------|--------------|-----------|
| Over | | Vector space | Dimension |
| 1. C | C | No | - |
| 2. R | R | No | - |
| 3. Q | Q | Yes | m.n |
| 4. Q^c | Q^c | No | - |
| 5. $Q(\sqrt{2})$ | $Q(\sqrt{2})$ | No | - |

| $Q(\sqrt{2})^{n \times m}$ | | | |
|----------------------------|---------------|--------------|--------------|
| Over | | Vector space | Dimension |
| 1. | C | No | - |
| 2. | R | No | - |
| 3. | Q | Yes | $2m \cdot n$ |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | $m \cdot n$ |

5. Let F' is subfield of F and let $F[x] = \{P(x) \mid P(x) \text{ is a polynomial with coefficients from field } F\}$, then $F[x]$ is Vector Space over Field F' under addition of polynomials and Scalar multiplication

| $C[x]$ | | | |
|--------|---------------|--------------|--------------------|
| Over | | Vector space | Dimension |
| 1. | C | Yes | Infinite dimension |
| 2. | R | Yes | Infinite dimension |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

| $R[x]$ | | | |
|--------|---------------|--------------|--------------------|
| Over | | Vector space | Dimension |
| 1. | C | No | - |
| 2. | R | Yes | Infinite dimension |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

| $Q[x]$ | | | |
|--------|-------|--------------|--------------------|
| Over | | Vector space | Dimension |
| 1. | C | No | - |
| 2. | R | No | - |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |

6. Let F' is subfield of F and $I = [0,1]$ be an interval containing 0 , let $F(I) = \{f \mid f. \text{ is a function from } I \text{ to } F'\}$, then $F(I)$ is Vector Space over Field F' under addition of functions and Scalar multiplication.
7. Which of the following is a Vector space under addition of functions and scalar multiplication of functions

| $C(I)$ | | | |
|--------|---------------|--------------|--------------------|
| | Over | Vector space | Dimension |
| 1. | C | Yes | Infinite dimension |
| 2. | R | Yes | Infinite dimension |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

| $R(I)$ | | | |
|--------|---------------|--------------|--------------------|
| | Over | Vector space | Dimension |
| 1. | C | No | - |
| 2. | R | Yes | Infinite dimension |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

| $Q(I)$ | | | |
|--------|---------------|--------------|--------------------|
| | Over | Vector space | Dimension |
| 1. | C | No | - |
| 2. | R | No | - |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | No | - |

| $Q(\sqrt{2})(I)$ | | | |
|------------------|---------------|--------------|--------------------|
| | Over | Vector space | Dimension |
| 1. | C | No | - |
| 2. | R | No | - |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

8. Let F' is subfield of F let $F^{(a_n)} = \langle a_n \rangle_n$ is a sequence on F' , then $F^{(a_n)}$ is Vector Space over Field F' under addition of sequence and Scalar multiplication
9. Which of the following is a Vector space under addition of sequence and scalar multiplication of sequence

| $C^{(a_n)}$ | | | |
|-------------|------|--------------|--------------------|
| | Over | Vector Space | Dimension |
| 1. | C | Yes | Infinite dimension |
| 2. | R | Yes | Infinite dimension |

| | | | |
|----|---------------|-----|--------------------|
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

| $R^{(a_n)}$ | | | |
|-------------|---------------|--------------|--------------------|
| | Over | Vector Space | Dimension |
| 1. | C | No | - |
| 2. | R | Yes | Infinite dimension |
| 3. | Q | Y_{es} | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Y_{es} | Infinite dimension |

| $Q^{<a_n>}$ | | | |
|-------------|---------------|--------------|--------------------|
| | Over | Vector space | Dimension |
| 1. | C | No | - |
| 2. | R | No | - |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | No | - |

| $Q(\sqrt{2})^{(a_n)}$ | | | |
|-----------------------|---------------|--------------|--------------------|
| | Over | Vector space | Dimension |
| 1. | C | No | - |
| 2. | R | No | - |
| 3. | Q | Yes | Infinite dimension |
| 4. | Q^c | No | - |
| 5. | $Q(\sqrt{2})$ | Yes | Infinite dimension |

Subspaces Spaces and Their Dimension

1. Which of the following is a subspace of given Vector space

| Vector space | | $C(C)$ | |
|--------------|------------------|----------|-----------|
| | Subset | Subspace | Dimension |
| 1. | $C(C)$ | Yes | 1 |
| 2. | $R(C)$ | No | - |
| 3. | $Q(C)$ | No | - |
| 4. | $Q(\sqrt{2})(C)$ | No | - |
| 5. | $C(R)$ | No | - |

| Vector space | | $C(R)$ | |
|--------------|------------------|----------|-----------------------|
| | Subset | Subspace | Dimension of subspace |
| 1. | $Q(R)$ | No | - |
| 2. | $R(Q)$ | No | - |
| 3. | $Q(\sqrt{2})(R)$ | No | - |
| 4. | $Q(\sqrt{2})(Q)$ | No | - |
| 5. | $R(R)$ | Yes | 1 |

| Vector space | | R(R) | |
|--------------|------------------|----------|-----------------------|
| Subset | | Subspace | Dimension of subspace |
| 1. | $C(R)$ | No | - |
| 2. | $R(R)$ | Yes | - |
| 3. | $Q(\sqrt{2})(R)$ | No | - |
| 4. | $Q(R)$ | No | - |

2. Which of the following is a subspace of given Vector space Let F' is subfield of F and let $F^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in F\}$ set of n -tuple's of field then

| Vector space | | $C^n(C)$ | |
|---|--|----------|---------------------------------|
| Subset | | Subspace | Dimension |
| $W_1 = \{(a_1, a_2, \dots, a_n) \mid \sum_{i=1}^n a_i = 0\}$ | | Yes | $n-1$ |
| $W_2 = \{(a_1, a_2, \dots, a_n) \mid \sum_{i=1}^n a_i a_i = 0, a_i \in F'\}$ | | Yes | $n-1$ |
| $W_3 = \{(a_1, a_2, \dots, a_n) \mid \sum_{i=1}^n i a_i = 0\}$ | | Yes | $n-1$ |
| $W_4 = \{(a_1, a_2, \dots, a_n) \mid a_{2i} = 0\} i = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor$ | | Yes | $\lfloor \frac{n+1}{2} \rfloor$ |
| $W_5 = \{(a_1, a_2, \dots, a_n) \mid a_{2i-1} = 0\} i = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor$ | | Yes | $\lfloor \frac{n}{2} \rfloor$ |
| $W_6 = \{(a_1, a_2, \dots, a_n) \mid a_i = 0 \text{ if } i \mid n\} i = 1, 2, \dots, n$ | | Yes | $n - \tau(n)$ |
| $W_7 = \{(a_1, a_2, \dots, a_n) \mid a_k \geq 0, 1 \leq k \leq n\}$ | | No | - |
| $W_8 = \{(a_1, a_2, \dots, a_n) \mid a_i = \lambda a_n, \lambda \text{ fixed}, \lambda \in F\}$ | | Yes | $n - 1$ |
| $W_9 = \{(a_1, a_2, \dots, a_n) \mid \frac{a_1}{a_n} = \lambda \text{ fixed}, \lambda \in F\}$ | | No | - |
| $W_{10} = \{(a_1, a_2, \dots, a_n) \mid \sum_{i=1}^n (-1)^{i+1} a_i = 0\}$ | | Yes | $n - 1$ |

| Vector Space | | $C^n(R)$ | |
|---|--|----------|-----------------------------------|
| Subset | | Subspace | Dimension |
| $W_1 = \{(a_1, a_2, \dots, a_n) \mid \sum_{i=1}^n a_i = 0\}$ | | Yes | $2(n-1)$ |
| $W_2 = \{(a_1, a_2, \dots, a_n) \mid \sum_{i=1}^n a_i a_i = 0, a_i \in F\}$ | | Yes | $2(n-1)$ |
| $W_3 = \{(a_1, a_2, \dots, a_n) \mid \sum_{i=1}^n i a_i = 0\}$ | | Yes | $2(n-1)$ |
| $W_4 = \{(a_1, a_2, \dots, a_n) \mid a_{2i} = 0\} i = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor$ | | Yes | $2 \lfloor \frac{n+1}{2} \rfloor$ |
| $W_5 = \{(a_1, a_2, \dots, a_n) \mid a_{2i-1} = 0\} i = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor$ | | Yes | $2 \lfloor \frac{n}{2} \rfloor$ |
| $W_6 = \{(a_1, a_2, \dots, a_n) \mid a_i = 0 \text{ if } i \mid n\} i = 1, 2, \dots, n$ | | Yes | $2(n - \tau(b))$ |
| $W_7 = \{(a_1, a_2, \dots, a_n) : a_k \geq 0, 1 \leq k \leq n\}$ | | No | - |
| $W_g = \{(a_1, a_2, \dots, a_n) \mid a_i = \lambda a_n, \lambda \text{ fixed}, \lambda \in F\}$ | | Yes | $2(n-1)$ |
| $W_9 = \{(a_1, a_2, \dots, a_n) \mid \frac{a_i}{a_n} = \lambda \text{ fixed}, \lambda \in F\}$ | | No | - |
| $W_{10} = \{(a_1, a_2, \dots, a_n) \mid \sum_{i=1}^n (-1)^{i+1} a_i = 0\}$ | | Yes | $2(n-1)$ |

3. Which of the following is a subspace of given Vector space of $F^{m \times n} = \{A = [a_{ij}] \mid a_{ij} \in F\}$ set of $m \times n$ matrices over field F . $F^{m \times n}(F')$ is Vector Space over Field F' under matrix addition and Scalar multiplication, where F' is subfield of F

4. Which of the following is a subspace of given Vector space $F^{m \times m}(F')$

| Vector Space | $C^{m \times m}(C)$ | |
|---|---------------------|------------------------|
| Subset | Subspace | Dimension |
| $W_1 = \{[a_{ij}] \mid \sum_{j=1}^m a_{ij} = 0, i = 1, 2, \dots, n\}$ | Yes | $mn - n$ |
| $W_2 = \{[a_{ij}] \mid \sum_{j=1}^m a_{ij} = 0, i = 1, 2, \dots, m\}$ | Yes | $mn - m$ |
| $W_3 = W_1 \cap W_2$ | Yes | $(n - 1)(m - 1)$ |
| $W_4 = W_1 \cup W_2$ | No | - |
| $W_5 = \{[a_{ij}] \mid a_{ij} = 0 \text{ if } i \mid n\}$ | Yes | $mn - m \cdot \tau(n)$ |

| Vector space | $C^{m \times m}(R)$ | |
|---|---------------------|---------------------------|
| Subset | Subspace | Dimension |
| $W_1 = \{[a_{ij}] \mid \sum_{j=1}^m a_{ij} = 0, i = 1, 2, \dots, n\}$ | Yes | $2(mn - n)$ |
| $W_2 = \{[a_{ij}] \mid \sum_{j=1}^m a_{ij} = 0, i = 1, 2, \dots, m\}$ | Yes | $2(mn - m)$ |
| $W_3 = W_1 \cap W_2$ | Yes | $2(n - 1)(m - 1)$ |
| $W_4 = W_1 \cup W_2$ | No | - |
| $W_5 = \{[a_{ij}] \mid a_{ij} = 0 \text{ if } i \mid n\}$ | Yes | $2(mn - m \cdot \tau(n))$ |

5. Which of the following is a subspace of given Vector space of $F^{n \times n} = \{A = [a_{ij}] \mid a_{ij} \in F\}$ set of $n \times n$ matrices over field F . $F^{n \times n}(F')$ is Vector Space over Field F' under matrix addition and Scalar multiplication, where F' is subfield of F

| Vector Space | $C^{n \times n}(C)$ | |
|--|---------------------|----------------------|
| Subset | Subspace | Dimension |
| $W_6 = \{A = [a_{ij}]_{n \times n} / \text{Trace}(A) = 0\}$ | Yes | $n^2 - 1$ |
| $W_7 = \{A = [a_{ij}]_{n \times n} / A^T = A\}$ | Yes | $\frac{n(n + 1)}{2}$ |
| $W_8 = \{A = [a_{ij}]_{n \times n} / A^T = -A\}$ | Yes | $\frac{n(n - 1)}{2}$ |
| $W_9 = W_6 \cap W_7$ | Yes | 0 |
| $W_{10} = W_6 \cup W_8$ | No | |
| $W_{11} = \{A = [a_{ij}]_{n \times n} / A \text{ is upper triangle}\}$ | Yes | $\frac{n(n + 1)}{2}$ |
| $W_{13} = \{A = [a_{ij}]_{n \times n} / A \text{ is diagonal}\}$ | Yes | n |
| $W_{14} = \{A = [a_{ij}]_{n \times n} / A \text{ is Scalar}\}$ | Yes | 1 |
| $W_{15} = \{A = [a_{ij}]_{n \times n} / A \text{ is tri diagonal}\}$ | Yes | $3n - 1$ |

| Vector Space | $C^{n \times n}(R)$ | |
|---|---------------------|--------------|
| Subset | Subspace | Dimension |
| $W_6 = \{A = [a_{ij}]_{n \times n} / \text{Trace}(A) = 0\}$ | Yes | $2(n^2 - 1)$ |
| $W_7 = \{A = [a_{ij}]_{n \times n} / A^T = A\}$ | Yes | $n(n + 1)$ |
| $W_8 = \{A = [a_{ij}]_{n \times n} / A^T = -A\}$ | Yes | $n(n - 1)$ |
| $W_9 = W_6 \cap W_7$ | Yes | 0 |

| | | |
|--|-----|----------|
| $W_{10} = W_6 \cup W_8$ | No | - |
| $W_{11} = \{A = [a_{ij}]_{n \times n} / A \text{ is upper triangle}\}$ | Yes | $n(n+1)$ |
| $W_{13} = \{A = [a_{ij}]_{n \times n} / A \text{ is diagonal}\}$ | Yes | $2n$ |
| $W_{14} = \{A = [a_{ij}]_{n \times n} / A \text{ is scalar}\}$ | Yes | 2 |
| $W_{15} = \{A = [a_{ij}]_{n \times n} / A \text{ is tri diagonal}\}$ | Yes | $6n - 2$ |

6. Which of the following is a subspace of given Vector space of $F[x] = \{P(x) \mid P(x) \text{ is a polynomial with coefficient from field } F\}$, then $F[x]$ is Vector Space over Field F' under addition of polynomials and Scalar multiplication, where F' is subfield of F

| Vector Space | $F[x]$ | |
|---|----------|-----------|
| | Subspace | Dimension |
| $W_1 = \{p(x) / \deg(p(x)) \leq n\}$ | Yes | $n+1$ |
| $W_2 = \{p(x) / p(x) \text{ is polynomial of even deg}\}$ | No | - |
| $W_3 = \{p(x) \mid p(x) \text{ zero polynomial or } \deg(p(x)) \geq k, k \text{ is a fixed natrued number}\}$ | No | - |
| $W_4 = \{p(x) / p(x) \text{ polynomial of deg } n\}$ | No | - |

7. Which of the following is a subspace of given Vector space of $F_n[x] = \{P_n(x) \mid P_n(x) \text{ where } P_n(x) \text{ is a polynomials of degree at most } n \text{ with coefficient from field } F\}$, then $F_n[x]$ is Vector Space over Field F' under addition of polynomials and Scalar multiplication, where F' is subfield of F

| Vector space | $R[x]$ | |
|--|----------|--------------------------------|
| | Subspace | Dimension |
| $W_5 = \{p(x) \in F_n[x] / p(a) = 0, a \in F\}$ | Yes | n |
| $W_6 = \{p(x) \in F_n[x] / p^k(a) = 0, p^{k+1}(x)\}$ | No | - |
| $W_7 = \{p(x) \in F_n[x] / p(x) = p(1-x)\}$ | yes | $\left[\frac{n}{2}\right] + 1$ |
| $W_8 = \{p(x) \in F_n[x] / p(a) = p(\beta), a \neq \beta \in F(\text{fixed})\}$ | Yes | n |
| $W_9 = \{p(x) \in F_n[x] / p(0) = 0\}$ | Yes | n |
| $W_{10} = \{p(x) \in F_n[x] / p(0) = 1\}$ | No | - |
| $W_{11} = \{p(x) \in F_n[x] / p(x) = p(-x)\}$ | Yes | $\left[\frac{n}{2}\right]$ |
| $W_{12} = \{p(x) \in F_n[x] / p(1) \geq 0\}$ | No | - |
| $W_{13} = \{p(x) \in F_n[x] / p'(0) = 0\}$, where $p'(x)$ is derivative of $p(x)$ | Yes | n |
| $W_{14} = \{p(x) \in F_n[x] / p(x) \text{ has a root in interval } [-1,1]\}$ | No | - |

Linear Transformation And Its Properties

Definition of Linear Transformation, Range and Null Space, Kinds of Linear Transformation, Representation of Transformation by Matrices, Linear Functional, Dual Space, Dual Basis, Annihilators,

Definitions

Linear Transformation: Let $V(F)$ and $V'(F)$ be vector spaces (over F). We call a function $T; V \rightarrow V'$ a linear transformation from $V(F)$ to $V'(F)$ if, for all $x, y \in V$ and $c \in F$, we have

- $T(x+y) = T(x) + T(y)$ and
- $T(cx) = cT(x)$

Example:

$R^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in R\}$ be a vector space over field R then

1. $T: R^n \rightarrow R^n$ such that $T(a_1, a_2, \dots, a_n) = (a_n, a_{n-1}, \dots, a_1)$
2. $T: R^n \rightarrow R^n$ such that $T(a_1, a_2, \dots, a_n) = (a_1 + a_2, a_2 + a_3, \dots, a_n + a_1)$
3. $T: R^n \rightarrow R^n$ such that $T(a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, a_{n-1}, 0)$
4. $T: R^n \rightarrow R^n$ such that $T(a_1, a_2, \dots, a_n) = (0, a_1, \dots, a_{n-1})$

Example:

$R^{n \times n} = \{A = [a_{ij}] \mid a_{ij} \in R\}$ be a vector space over field R then

1. $T: R^{n \times n} \rightarrow R^{n \times n}$ such that $T(A) = A^T$: where A^T denotes the transpose of A
2. $T: R^{n \times n} \rightarrow R^{n \times n}$ such that $T(A) = \frac{A+A^T}{2}$: where A^T denotes the transpose of A
3. $T: R^{n \times n} \rightarrow R$ such that $T(A) = Tr(A)$ where $Tr(A)$ denotes the Trace of A
4. $T: R^{n \times n} \rightarrow R$ such that $T(A) = \sum_{j=1}^n \sum_{i=1}^n a_{ij}$ where $A = [a_{ij}]$ denotes the Trace of A

Example:

$R[x] = \{p(x) \mid p(x) \text{ is polynomial space over field } R\}$ then

1. $T: R[x] \rightarrow R[x]$ such that $T(p(x)) = p'(x)$: where $p'(x)$ denotes the derivative of $p(x)$
2. $T: R[x] \rightarrow R[x]$ such that $T(p(x)) = p''(x)$: where $p''(x)$ denotes the second derivative of $p(x)$
3. $T: R[x] \rightarrow R[x]$ such that $T(p(x)) = \int_0^x p(t) dt$
4. $T: R[x] \rightarrow R$ such that $T(p(x)) = p(0)$
5. $T: R[x] \rightarrow R$ such that $T(p(x)) = p'(0)$

Example:

Let V be a vector space over field F

1. Identity transformation $I(x) = x$ is linear transformation from V onto V
2. Zero transformation 0 defined by $O(x) = 0$ called trivial transformation

Let $V(F)$ and $V'(F)$ be vector spaces (over F), let $T: V \rightarrow V'$ be a linear transformation from $V(F)$ to $V'(F)$

Range of A Linear Transformation T : Range of T is $R(T) = \{T(x) \mid x \in V\}$

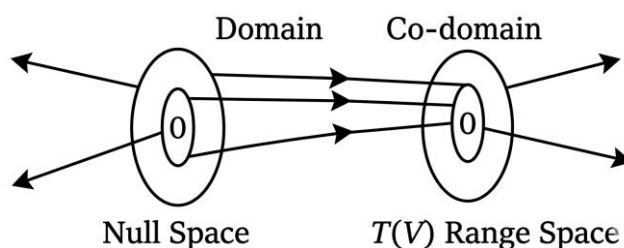
Proposition 2.1: Range of a linear transformation $T: V \rightarrow V'$ is subspace of V' and called range space of linear transformation

Rank of A Linear Transformation T : The dimension of range space of $R(T)$ of a linear transformation called Rank of a linear transformation T denoted by $\rho(T)$.

Kernel of A Linear Transformation T : kernel of T is $ker(T) = \{x \in V \mid T(x) = 0\}$ set of all vectors which maps to zero of V' .

Proposition 2.2: kernel of a linear transformation $T: V \rightarrow V'$ is subspace of V and called null space of linear transformation T .

Nullity of a linear transformation T : The dimension of null space of $N(T)$ or $ker(T)$ of a linear transformation called Nullity of a linear transformation T denoted by $\eta(T)$



Sylvester's Law: Let V and W be vector spaces over the field F and let T be a linear transformation from V into W . Suppose that V is finite dimension then $\rho(T) + \eta(T) = \dim(V)$

Proposition 2.3: Let T be a linear operator on a finite dimensional vector space V . Suppose that $\text{rank}(T^2) = \text{rank}(T)$, the range and the null space of T have only zero vector in common i.e. $\text{Range } T \cap \text{Ker } T = \{0\}$

Kinds of Linear Transformation

Linear Operator: Let $V(F)$ be a vector space than any linear transformation from V into V is called linear operator on V .

Singular: A linear transformation called singular if there exists a nonzero vector x such that $T(x) = 0$ that is a linear transformation is singular if $\text{ker}(T) \neq 0$ that is $\eta(T) \geq 1$

Non-Singular: A linear transformation is called non-singular if $T(x) = 0 \Rightarrow x = 0$ i.e., a linear transformation is non-singular if $\text{ker}(T) = \{0\}$

Proposition 2.4: T is non-singular $\Leftrightarrow T$ is one-one $\Leftrightarrow \text{ker}(T) = \{0\}$

Invertible Linear Transformation: Let $V(F)$ and $V'(F)$ be vector spaces over the field F and let $T: V \rightarrow V'$ be a linear transformation such that T is one-one and onto then T is called invertible.

In other words, a linear transformation $T: V \rightarrow V'$ is invertible iff \exists a linear transformation $S: V' \rightarrow V$ such that $TS = I = ST$

Proposition 2.5: Let V, W and Z be vector spaces over the field F . Let T be a linear transformation from V into W and S is a linear transformation from W into Z . then the composite function SoT defined by $(SoT)(x) = S(T(x)) \forall x \in V$ is linear transformation from V into Z .

Proposition 2.6: Let T be a linear transformation from V into W , then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W .

Proposition 2.7: Let V and W be two finite dimensional vector space such that $\dim V = \dim W$ and $T: V \rightarrow W$ a linear transformation. Then the following conditions are equivalent:

1. T is invertible
2. T is non-singular
3. T is onto
4. If $\{v_1, v_2, \dots, v_n\}$ is a basis of V , then $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis of W .

Proposition 2.8: if $T: V \rightarrow W$ is a linear transformation such that $\dim V = \dim W$. Then T is invertible $\Leftrightarrow T$ is 1 - 1 $\Leftrightarrow T$ is non singular $\Leftrightarrow T$ is onto

Proposition 2.9: Let V and W be vector spaces over the field F and let T be a linear transformation from V into W . If T is invertible, then the inverse function T^{-1} is a linear transformation from W onto V

Proposition 2.10: If T and U are linear transformations on a finite dimensional vector space V such that $TU = I$, then T and U are invertible and $T^{-1} = U$.

Note: Above can be false when V is not finite dimensional.

Some the Important Linear Operators

Projection: Let $V = R^2$ be a vector space over field R consider the function $T: R^2 \rightarrow R^2$

1. T is projection on x -axis alone y -axis then $T(a, b) = (a, 0)$ for all $(a, b) \in R^2$ is linear operator
2. T is projection on y -axis alone x -axis then $T(a, b) = (0, b)$ for all $(a, b) \in R^2$ is linear operator
3. T is projection on $y = mx$ axis alone $y = -\frac{1}{m}x$ axis then

$$T(a, b) = \left(\frac{a + mb}{1 + m^2}, \frac{m(a + mb)}{1 + m^2} \right) \text{ for all } (a, b) \in R^2 \text{ is linear operator}$$

Orthogonal Projection on Subspace: Let $V(F)$ be a vector space over field F and let W be subspace of $V(F)$, define

$T: V \rightarrow V$ such that $T(w + w') = w$, where $V = W \oplus W^1$, (W^1 is orthogonal complement subspace of W) then

1. T is a linear operator V , is called orthogonal projection on subspace w
2. $Ker(T) = W^1$ and $Range(T) = W$
3. $\eta(T) = \dim W^1$ and $\rho(T) = \dim W$
4. If $W \neq V$ then T neither one-one nor onto, i.e., T is singular linear operator
5. $T^2 = T$, i.e., T is idempotent linear operator

Reflection: Let $V = R^2$ be a vector space over field R consider the function $T: R^2 \rightarrow R^2$

1. T is reflection about x -axis alone y -axis then $T(a, b) = (a, -b)$ for all $(a, b) \in R^2$ is linear operator
2. T is reflection about y -axis alone x -axis then $T(a, b) = (-a, b)$ for all $(a, b) \in R^2$ is linear operator
3. T is reflection about line $y = mx$, alone $y = -\frac{1}{m}x$ then $T(a, b) = \left(2\left(\frac{a+mb}{1+m^2}\right) - a, 2\left(\frac{m(a+mb)}{1+m^2}\right) - b\right)$ for all $(a, b) \in R^2$ is linear operator
4. T is reflection about origin then $T(a, b) = (-a, -b)$ for all $(a, b) \in R^2$ is linear Operator

Reflection on Subspace Along its Orthogonal Complement:

Let $V(F)$ be a vector space over field F and let W be subspace of $V(F)$, define

$T: V \rightarrow V$ such that $T(w + w') = w - w'$, where $V = W \oplus W'$, (W' is orthogonal complement subspace of W) then

1. T is a linear operator V , is called reflection about subspace W along W' is orthogonal complement subspace of W
2. $Ker(T) = \{0\}$ and $Range(T) = V$
3. $\eta(T) = 0$ and $\rho(T) = \dim V$
4. T one-one and onto, i.e., T is invertible linear operator
5. $T^2 = I$, i.e., T is Involuntary linear operator

Rotation: Let $V = R^2$ be a vector space over field R consider the function $T: R^2 \rightarrow R^2$, T is rotation in anti-clock wise by an angle θ then $T(a, b) = (a \cos \theta - b \sin \theta, a \sin \theta + b \cos \theta)$ for all $(a, b) \in R^2$ is a linear operator .

1. $Ker(T) = \{0\}$ and $Range(T) = V$
2. $\eta(T) = 0$ and $\rho(T) = \dim V$
3. T one-one and onto, i.e., T is Invertible linear operator
4. If T_θ & T_ϕ are rotation in anti-clock wise by an angle θ and ϕ then $T_\theta \circ T_\phi$ rotation in anti-clock wise by an angle $\theta + \phi$
5. $G = \{T_\theta: R^2 \rightarrow R^2 \mid (\theta)_n = 360\theta T_\theta \circ T_{\theta_1} \circ \dots \circ T_\theta = T_{n\theta} \text{ are rotation by an angle } \theta \}$ then under Composition of function G be infinite order abelian group
6. G are rotation by an angle θ then $T_\theta^n = T_\theta \circ T_\theta \circ \dots \circ T_\theta = T_{n\theta}$
7. G are rotation by an angle θ then there exists $T_\theta^n = I$ for some $n \in N$ iff θ is rational multiple of π
8. If θ is rational multiple of π and G are rotation by an angle θ such that $T_\theta^n = I$ then $\theta = \left(\frac{360^\circ}{n}\right)$

Let $V = R^3$ be a vector space over field R consider the function $T: R^3 \rightarrow R^3$,

1. T is rotation in anti-clock wise by an angle θ in xy -plane then $T(a, b, c) = (a \cos \theta - b \sin \theta, a \sin \theta + b \cos \theta, c)$ for all $(a, b, c) \in R^3$ is a linear operator .