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Mathematical Science

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SIERRA INNOVATIONS

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1

CHAPTER

Complex Analysis

Fundamental Concepts of Complex Analysis

Introduction

In the field of real numbers, the equation $x^2 + 1 = 0$ has no solution. To permit the solution of this and similar equations, the real number system was extended to the set of complex numbers. Euler introduced the symbol i with property that $i^2 = -1$. He also called i is the imaginary unit. A number of the form $a + ib$, where a, b are real numbers, was called complex number.

If we write $z = x + iy$, then z is called a complex variable. Also x and y are respectively called real and imaginary parts of z. Sometimes we express z as $z = (x, y)$.

we also write

$Re(z) = x, Im(z) = y$.

If $x = 0$, i.e., $z = iy$, the z is called purely imaginary number.

If $y = 0$ i.e., $z = x$ then z is called purely real number.

The complex conjugate, or briefly conjugate, of $z = x + iy$ is $z^{-} = x - iy$.

For example conjugate of $-3 - 5i$ is $-3 + 5i$.

It is easy to verify that

$$Re(z) = x = \frac{z + z^{-}}{2}, Im(z) = y = \frac{z - z^{-}}{2i}$$

Geometrical Representation of Complex Numbers

Consider the complex number $z = x + iy$.

A complex number can be regarded as an ordered pair of reals i.e. $z = (x, y)$.

This form of z suggests that z can be represented by a point P whose co-ordinates are x and y relative to rectangular axes X and Y.

To each complex number there corresponds one and only one point in the xy-plane and conversely to each point in the plane there exists one and only one complex number. Due to this fact, the complex number z is referred to the point z in this plane. This plane is called complex plane or Gaussian plane or Argand plane. The representations of complex numbers is called Argand diagram. The complex number $x + iy$ is called complex co-ordinate; and x and y axes are respectively called real and imaginary axes. The distance between the points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, is

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The complex number z is called affix of the point (x, y)

Polar Forms of a Complex Numbers $P(x, y)$

Consider a point P in the complex plane corresponding to a complex number $z = x + iy$. From the adjoining figure.

$$x = r\cos\,\theta, y = r\sin\,\theta$$

Then $r = \sqrt{x^2 + y^2} = |x + iy| = |z|$

$\theta = tan^{-1}\left(\frac{y}{x}\right)$.

It follows that $z = x + iy$

$= r(cos\ \theta + isin\ \theta) = re^{i\theta}$ ... (1)

It is called polar form of the complex number z.

r and θ are called polar co-ordinates of z. r is called modulus or absolute value of z.

(Modulus of any complex number z is equal to distance of that point from the origin). The angle θ which the line OP makes with the positive x-axis is called argument or amplitude of z. It is also written as $\theta = amp(z)$ or $\theta = arg\ (z)$.

The argument of z is not unique, since the equation (1) does not alter, if we replace θ by $2\pi + \theta$. So θ can have infinite number of values which differ from each other by 2π. In order to specify a unique value of arg(z), we may restrict its value to some interval of length 2π. For this we introduce the concept of "principal value" of arg(z) as follows:

For an arbitrary $z \neq 0$, the principal value of Arg z is defined to be the unique value of z that satisfies $-\pi < arg\ z \leq \pi$ or $-\theta \leq \pi$ and it will be denoted by arg z. Thus the relation between Arg z and $arg\ z$ is given by

$$arg\ z = Argz + 2k\pi; k = 0, \pm 1, \pm 2, \dots \dots$$

For convenience the set of all the values of $Argz$ is denoted by *arg z. In fact while inverting second equation in we should note the following :

$$\text{Arg}z = \begin{cases} \text{Arctan}\left(\frac{y}{x}\right) & \text{if } x > 0, y \geq 0 \\ \pi + \text{Arctan}\left(\frac{y}{x}\right) & \text{if } x > 0, y > 0 \\ & \text{and } x < 0, y \geq 0 \\ -\pi + \text{Arctan}\left(\frac{y}{x}\right) & \text{if } x < 0, y < 0, \\ \pi/2 & \text{if } x = 0, y > 0 \\ -\pi/2 & \text{if } x = 0, y < 0 \end{cases}$$

Where Arctan X is the principal value of the arctangent of a real number X, satisfying the inequality $-\frac{\pi}{2} < arctan\ X \leq \frac{\pi}{2}$.

Results:

(i) The modulus of the product of two complex numbers is the product of their moduli.

$|z_1 z_2| = |z_1||z_2|$

(ii) The modulus of the sum of two complex numbers is less than or equal to sum of their moduli.

$|z_1 + z_2| \leq |z_1| + |z_2|$

(iii) The modulus of the difference of two complex numbers is greater than or equal to the difference of their moduli.

$|z_1 - z_2| \geq |z_1| - |z_2|$

(iv) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

(v) Equation of a circle the equation of a circle in the Argand plane can be put in the form.

$$zz^{-} + bz^{-} + b^{-}z + c = 0$$

where c is real and b is complex constant. If $b = \alpha + i\beta$ then centre of the circle is given by $(-\alpha, -\beta)$& radius $= \sqrt{\alpha^2 + \beta^2 - c}$

Inverse Points w. r. t. A Circle

Two points $A(z=a)$ and $B(z=b)$ are said to be inverse points of a circle with centre $O(z=z_0)$ and radius r if O, A, B are collinear and $O.A.OB = r^2$.

So that $|a-z_0||b-z_0| = r^2$

Then $arg\ (a-z_0) = arg\ (b-z_0) = -arg\ (b-z_0)$

This gives are $(a-z_0) + arg\ (b-z_0) = 0$

This $\Rightarrow (a-z_0)(b-z_0)$ is real and equal to r^2.

Hence $\Rightarrow (a-z_0)(b-z_0) = r^2$ is the required equation.

Deduction:

(i) If $z_0 = 0$, then a and b are inverse point if $\underline{ab} = r^2$.

Hence z_1, z_2 are inverse points of the circle $|z| = r$ if $z_1\underline{z_2} = r^2$.

Finally $z_1, \frac{1}{z_1}$ are inverse point of each other w. r. t. $|z| = r$.

(ii) $z = 0, z = \infty$ are inverse point of each other w. r.t. $|z| = r$.

Example: Show that inverse point of a point $z = a$ w. r. t. circle $|z-c| = r$ is the point $c + \frac{r^2}{a-c}$.

Solution: $|z-c| = r$ … (1)

Let $B(b)$ is the inverse for $A(a)$ w. r. t. circle (1) whose centre is $C(c)$ and radius r.

Then following conditions hold

(i) $CB.CA = r^2$

(ii) Lines CA and CB are collinear

From above

(i) $\Rightarrow |b-c||a-c| = r^2$

(ii) $\Rightarrow arg\ (b-c) = arg\ (a-c) = -arg\ (\bar{a}-\bar{c})$. { For $arg\ z = -arg\ \bar{z}$}

Or, $arg\ [(b-c)(\bar{a}-\bar{b})] = 0$

For $arg\ (z_1 z_2) = arg\ z_1 + arg\ (z_2)$

This $\Rightarrow (b-c)(\bar{a}-\bar{c})$ is real and positive

$$\Rightarrow |(b-c)(\bar{a}-\bar{c})| \Rightarrow (b-c)(\bar{a}-\bar{c}) > 0 \qquad \dots (2)$$

Now (1) $\Rightarrow |b-c||\bar{a}-\bar{c}| = r^2$

$$\Rightarrow |(b-c)(\bar{a}-\bar{c})| = r^2$$

Using (2) we get

$$|(b-c)(\bar{a}-\bar{c})| = r^2$$

Or $b = c + \frac{r^2}{\bar{a}-\bar{c}}$.

Stereographic Projection And Point Set Topology

We shall extend the complex plane C by adjoining one extra point at infinity which we shall denote by $z = \infty$, i.e., we consider the extended complex plane $C \cup \{\infty\}$ as a closed surface having a single point at infinity.

We shall then introduce a new metric to describe the behavior of a complex function at infinity and to map the points in C into the surface of a sphere. This process will be referred to as stereographic projection. In fact such an extra point ' ∞ ' is defined so as to satisfy the following computational properties.

Whatever be $z \in C$

$$\frac{z}{\infty}=0;\ z+\infty=\infty(z\neq\infty);\frac{z}{0}=\infty(z\neq 0);\ \ z\cdot\infty=\infty(z\neq 0);\frac{z}{\infty}=0(z\neq\infty).$$

We do not define $\frac{0}{0}, \infty+\infty, \infty-\infty$ and $\frac{\infty}{\infty}$.

We use the following construction due to Riemann. There are two commonly used methods, according as the complex plane C is the tangent to the sphere or passes through its centre. We shall use the first one. Now we set up a correspondence between the points of C and those of a sphere of radius $\frac{1}{2}$ with centre at $\left(0,0,\frac{1}{2}\right)$ tangent to this plane. (There is another method of correspondence in which the sphere of radius 1 has centre at (0,0,0) and the plane passes through (0,0,0).)

Let C be the complex plane. Through the origin construct a line perpendicular to C. Let this be ξ-axis of a 3 -dimensional Euclidean space in which a point has coordinates (ξ,η,ζ). Consider the sphere S of radius $\frac{1}{2}$ and centre at $\left(0,0,\frac{1}{2}\right)$. That is

$$S=\left\{(\xi,\eta,\zeta)\in R^3:\xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2=\frac{1}{4}\right\}$$

It is a common practice to call the points N and O with coordinates (0,0,1) and (0,0,0) the North Pole and South Pole of the sphere S respectively.

The great circle in the plane $\zeta=\frac{1}{2}$ is called the equator. The plane $\zeta=0$ coincides with the complex the x and y axes respectively. Let $Q(x,y,0)$ be any point in the plane C.

Through the points N and Q we draw a straight line NQ intersecting the sphere S at a point say $P(\xi,\eta,\zeta)$. Then (ξ,η,ζ) is called the stereographic projection, or image of (x,y,O) on the sphere and is considered as the spherical representation of $z=x+iy$. This procedure assigns a unique point on S to every given complex number z. Conversely to each point (ξ,η,ζ) on the sphere other than N we can associate the complex number $z=(x,y,0)$ where the line from (0,0,1) through (ξ,η,ζ) intersects C.

Now we immediately see that there is a one to one correspondence between C and the points of S with one exception, namely the north pole (0,0,1) itself. By assigning to the north pole N of the sphere to correspond to the point at infinity, we obtain then a one to one correspondence between the points of the sphere S on one hand and the point of the extended complex plane $C\cup\{\infty\}$ on the obtain explicit equationsexpressing ξ,η and ζ in terms of x and y. The line in R^3 passing through (0,0,1) and $(x,y,0)$ is given by

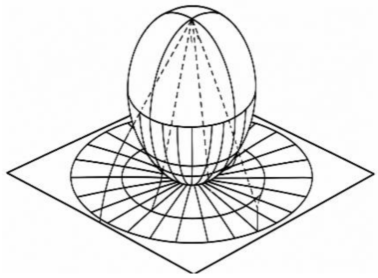
$$\{t(0,0,1)+(1-t)(x,y,0):t\in R\};$$

that is

$$\{((1-t)x,(1-t)y,t):t\in R\}$$

Since this line intersects the sphere S, we must have

$C\rightarrow\left(0,0,\frac{1}{2}\right)$



$O \to (0,0,0)$
$Q \to (x, y, 0)$

$$(1-t)^2x^2 + (1-t)^2y^2 + \left(t - \frac{1}{2}\right)^2 = \frac{1}{4}$$

So that $(1-t)^2|z|^2 = t(1-t)$.
If $(\xi, \eta, \zeta) \neq (0,0,1)$ we arrive at
$t = \frac{|z|^2}{1+|z|^2}$, i.e., $1 - t = \frac{1}{1+|z|^2}$
Using the fact that the points $(0,0,1), N \to (0,0,1)$
(ξ, η, ζ) and $(x, y, 0)$ are collinear now yields $P \to (\xi, \eta, \zeta)$

$$\left.\begin{aligned} \xi &= \frac{x}{1+x^2+y^2} = \frac{z+z}{2(1+|z|^2)}, \\ \eta &= \frac{y}{1+x^2+y^2} = \frac{-i(z-\bar{z})}{2(1+|z|^2)}, \\ \xi &= \frac{x^2+y^2}{1+x^2+y^2} = \frac{|z|^2}{1+|z|^2}; \end{aligned}\right\}$$

that is $z = x + iy \in C$ corresponds to $\equiv (\xi, \eta, 0)$

$$\left(\frac{x}{1+x^2+y^2}, \frac{y}{1+x^2+y^2}, \frac{x^2+y^2}{1+x^2+y^2}\right) \in C$$

$$Or \left(\frac{z+z^-}{2(1+|z|^2)}, \frac{-i(z-z^-)}{2(1+|z|^2)}, \frac{|z|^2}{1+|z|^2}\right) \in S$$

For instance, the images of $1, i, \frac{1-i}{\sqrt{2}}$ on the sphere are respectively given by z_1, z_2, z_3

Where $z_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$

$z_2 = \left(0, \frac{1}{2}, \frac{1}{2}\right)$

$z_3 = \left(\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, \frac{1}{2}\right)$

Using the three equations in it is easy to see that

$1 - \zeta = \frac{1}{1+|z|^2}$; (i.e., $|z|^2 = \frac{\zeta}{1-\zeta}$)

So that $x = \frac{\xi}{1-\zeta}, y = \frac{\eta}{1-\zeta}$;

Or $z = \frac{\xi + i\eta}{1-\zeta}$

That is (ξ, η, ζ) corresponds to $(\frac{\xi}{1-\zeta} + i\left(\frac{\eta}{1-\zeta}\right) \in C$

The map $z \leftrightarrow (\xi, \eta, \zeta)$ is called stereographic projection of C on $S \setminus \{(0,0,1)\}$ or vice versa.

Example: Prove that the points z and z' in the complex plane will represent symmetric points with respect to the equatorial plane (viz. the plane $\zeta = \frac{1}{2}$) if and only if $zz' = 1$

Solution: Note that z and z' correspond to symmetric points with respect to equatorial plane if and only if z corresponds to (ξ, η, ζ) and z' corresponds to $(\xi, \eta, 1-\zeta)$.
This holds if the only if

$$z = \frac{\xi + i\eta}{1-\zeta} \;\; and \;\; z' = \frac{\xi + i\eta}{1-(1-\zeta)} = \frac{\xi + i\eta}{\zeta}$$

i.e., if and only if $zz' = \frac{\xi^2+\eta^2}{(1-\zeta)\zeta} = 1$

Chordal Distance

Let z_1 and z_2 be the points in the argand plane corresponds steriographically to z_1 and z_2 respectively. Distance between z_1 and z_2 i.e. length of chord joining z_1 and z_2 is referred as chordal distance between z_1 and z_2

Example: Find the chordal distance $z_1 = 1 + i, z_2 = \infty \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$ (0,0,1)

$$\textbf{\textit{Distance}} = \sqrt{\left(\frac{1}{3}-0\right)^2 + \left(\frac{1}{3}-0\right)^2 + \left(\frac{2}{3}-1\right)^2} = \sqrt{\frac{1}{9}+\frac{1}{9}+\frac{1}{9}} = \frac{1}{\sqrt{3}}$$

Chordal Distance between z and ∞ is denoted by $\gamma(z, \infty)$ and is equal to

$$\chi(z, \infty) = \frac{1}{\sqrt{1+|z|^2}}$$

Chordal Distance between z_1 and z_2 is calculated by the following formula

$$\chi(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2} \times \sqrt{1+|z_2|^2}}$$

If $\chi(z_1, z_2) = 1$ then

$$\frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2} \times \sqrt{1+|z_2|^2}} = 1 \;\; |z_1 - z_2| = \sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}$$

Squaring both side

$(z_1 - z_2)\left(\underline{z_1} - \underline{z_2}\right) = (1+|z_1|^2)(1+|z_2|^2)$

$z_1\underline{z_1} - z_1\underline{z_2} - z_2\underline{z_1} + z_2\underline{z_2} = 1 + z_2\underline{z_2} + z_1\underline{z_1} + z_1z_2\underline{z_1z_2}$

$z_1\bar{z}_2 + z_2\bar{z}_1 + z_1z_2\bar{z}_1\bar{z}_2 + 1 = 0$

$z_1\bar{z}_2(z_2\bar{z}_1 + 1) + (z_2\bar{z}_1 + 1) = 0$

$(z_1\bar{z}_2 + 1)(z_2\bar{z}_1 + 1) = 0$

$z_1\bar{z}_2 = -1$

$z_2\bar{z}_1 = -1$

If on the sphere two point diametrically opposite then $z_1\bar{z}_2 = -1$

Sets of Points in the Complex Plane

Let S be a non-empty set of complex numbers and $z_0 \in S$ be any complex number and δ be a positive real number. Then, we define the following:

1. **Circle:** The set of points which satisfies the equation $|z - z_0| = \delta$ or

$$(x-x_0)^2 + (y-y_0)^2 = \delta^2$$

Defines a circle C of radius δ with centre at $z_0 = (x_0, y_0)$. This set consists of all points which lie on the boundary of the circle C. Any point z on this circle has the polar form $z = z_0 + \delta e^{i\theta}$. As θ varies from 0 to $2\pi, z$ traverses once over this circle in the counter clockwise direction. If $z_0 = 0$, then the equation $|z| = \delta$ defines a circle of radius δ about the origin.

2. **Open Disk:** An open disc is defined as $\Delta(z_0;\epsilon) = \{z \in C: |z - z_0| < \epsilon\}$ with center at the point z_0 and radius ϵ i.e. it is the collection of all those point inside the circle $|z - z_0| = \epsilon$.

3. **Closed Disk:** An open disc is defined as $\Delta(z_0;\epsilon) = \{z \in C: |z - z_0| \le \epsilon\}$ with center at the point z_0 and radius ϵ i.e. it is the collection of all those point inside and on the circle $|z - z_0| = \epsilon$.

4. **Annulus:** The set of points which lie between two concentric circles $C_1: |z - z_0| = r_1$ and $C_2: |z - z_0| = r_2$ defines an open annulus or an open circular ring, that is the set of points which satisfies the inequality $r_1 < |z - z_0| < r_2$.
The set of points which satisfies the inequality $r_1 \le |z - z_0| \le r_2$, defines a closed annulus.

5. **Neighborhood of A Point:** A δ-neighborhood of a point z_0 in the complex plane is the set of all points z which lie in the open disk $|z - z_0| < \delta$. Usually, the δ-neighborhood about the point z_0 is denoted by $N(z_0, \delta)$ or $N_6(z_0)$. If we exclude the point z_0 from the open disk $|z - z_0| < \delta$, then it is called the deleted neighborhood of the point z_0 and is written as $0 < |z - z_0| < \delta$.

6. **Interior Point:** A point z is an interior point of S, if for some s - 0 $N = s) \sim C$

7. **Exterior Point:** A point z is an exterior point of S, it is the interior of the set S^c.

8. **Frontier Point:** $a \in C$ is said to be frontier point of $S \subseteq C$, if it is neither an interior point of S nor an exterior point of S. Set of all frontier point of S is denoted by Fr S

9. **Boundary Point:** Frontier point of S which are member of S are referred as boundary point. Set of all boundary point of S is denoted by ∂S.
Example: For the set of points defined by $Im(z) \ge 1$, the points on the line $y = 1$ are the boundary points. The points on the circle $|z - z_0| = r$ are the boundary points for the disk $|z - z_0| \le r$. The collection of all the boundary points define the boundary of S.

10. **Open Set:** A set S is open, if every point of S is an interior point.
Example: The sets

$$S = \{z: |z - z_0| < r\}; S = \{z: Re(z) < 0\};$$

$S = \{z: |z| < 2\}$. are open sets.

11. **Closed Set:** A set S is closed, if every boundary point of S belongs to S.
Example: The sets $S = \{z: |z - z_0| \le r\}$;
$S = \{z: r_1 \le |z - z_0| \le r_2\}$ are closed sets.

12. **Bounded Set:** An open set S in bounded, if there exists a positive real number M, such that $|z| \le M$ for all $z \in S$. Otherwise, the set S is said to be unbounded.
Example: The set $S = \{z: |z - z_0| < r\}$ is a bounded and $S = \{z: |z - z_0| > 0\}$ is an unbounded.

13. **Connected Set:** A subset S of C is said to be connected if the only subsets of S which are both open and closed are ϕ and S.

14. **Domain:** An open connected set is called a domain. Usually, a domain is denoted by D.

15. **Region:** A region is a domain together with all, some or none of its boundary points. Thus, a domain is always a region but a region may or may not be a domain.
Example: An open disk is both domain and region but closed disk is region but not domain. Usually, a region is denoted by R.

16. Extended Complex Plane: The complex plane to which the point at $z = \infty$ has been added is called the extended complex plane. The complex plane without the point at $z = \infty$ is called the finite complex plane.

Limit Continuity & Differentiability

Limit of A Function

Let $w = f(z)$ be a complex valued function f defined on $D \subseteq C$ let $z_0 \in D'$. Then f is said to have a limit l as $z \to z_0$ and we write

$$\lim_{z\to z_0} f(z) = l \text{ or } f(z) \to l \text{ as } z \to z_0$$

if and only if for any given $\varepsilon > 0, \exists a\delta > 0$ such that $|f(z) - l| < \varepsilon$ whenever $z \in D \& 0 < |z - z_0| < \delta$; i.e., if and only if for each $\varepsilon > 0, \exists \delta > 0$ such that $f(z) \in B(l; \varepsilon)$ whenever $z \in B(z_0, \delta) \setminus \{z_0\} \cap D$. $z \to z_0$ in complex plane. It is straight forward to state

$$lim_{z\to z_0} f(z) = l \Leftrightarrow lim_{z\to z_0} |f(z) - l| = 0$$

Note:

(i) The function need not to be defined at z_0 in order to have a limit at z_0.

(ii) It is the punctured disk $B(z_0, \delta) \setminus \{z_0\}$ which is involved in D, i.e., z_0 need not to be in D.

(iii) If the condition that $z_0 \in D$ holds, we may have $f(z_0) \neq l$.

Alter Definition of Limit of A Function

Let $f(z) = u(z) + iv(z)$, where $u(z) = u(x, y)$ & $v(z) = v(x, y)$ are real valued functions, be defined on D except possibly at z_0. Then for $l_1 \& l_2 \in R$.

$$\lim_{z\to z_0} f(z) = l_1 + il_2$$

If and only if $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = l_1 \& \lim_{(x,y)\to(x_0,y_0)} v(x, y) = l_2$.

Theorem: If $\lim_{z\to z_0} f(z)$ exists, then it is unique.

Examples:

1. Let $z \neq 0$ consider $f(z) = \frac{\bar{z}}{z}$, now let us examine the limits of $f(z)$ as $z \to 0$ in many ways. Let m be any real number & allow $z \to 0$ along the line $y = mx$ then

$$\lim_{(x+imx)\to 0} \frac{x - imx}{x + imx} = \frac{1 - im}{1 + im}$$

$\Rightarrow \lim_{z\to 0} f(z)$ depends upon the path as we approach towards z_0 hence limit does not exist at $z = 0$.

2. Let $f(z) = x\sin\frac{1}{x} + iy\sin\frac{1}{y}$ then its domain of definition is the collection of all those points = which does not lie on x axis & y axis.

Consider $\lim_{z\to(0)(0)} f(z) = \lim_{x\to 0} x\sin\frac{1}{x} + i\lim_{y\to 0} y\sin\frac{1}{y}$

$= 0 + iy_0\sin\frac{1}{y_0}$

hence limit exist.

3. $lim_{z\to 0} z^m \sin\frac{1}{z}$ does not exist $\forall m \in N$.

4. $lim_{z\to 0} e^{-\frac{1}{z^m}}$ such that $m \in N$ does not exist.

Theorem: If $f(z)$ has a finite limit at z_0, then $f(z)$ is a bounded function in some neighborhood of z_0.

Theorem: Let $f(z) \& g(z)$ be two functions such that $lim_{z\to z_0} f(z) = l_1 \& lim_{z\to z_0} g(z) = l_2$

Then

(a) $lim_{z\to z_0}[\alpha f(z)] = \alpha l_1$ where α is a real or complex constant.

(b) $lim_{z\to z_0}[f(z) \pm g(z)] = l_1 \pm l_2$

(c) $lim_{z\to z_0}[f(z)\cdot g(z)] = l_1 \cdot l_2$

(d) $lim_{z\to z_0}\left[\frac{1}{g(z)}\right] = \frac{1}{l_2}, l_2 \neq 0$

(e) $lim_{z\to z_0}\left[\frac{f(z)}{g(z)}\right] = \frac{l_1}{l_2}, l_2 \neq 0$

Limit of a Function at $z = \infty$

The function $f(z)$ has a limit L as $z \to \infty$, if for any arbitrary small real number $\varepsilon > 0, \exists$ a real number $\delta > 0$ such that

$|f(z) - L| < \varepsilon$ whenever $|z| > \frac{1}{\delta}$

Alternately, we substitute $z = \frac{1}{\xi}$, since $\xi \to 0$ as $z \to \infty$ we obtain

$lim_{z\to\infty} f(z) = lim_{\xi\to 0} f\left(\frac{1}{\xi}\right)$

Continuous Function

A function $f: D \to C$ is continuous at $z_0 \in D$ if and only if $lim_{z\to z_0} f(z)$ exists & equals to the functional value $f(z_0)$. We say that f is continuous on D or $f: D \to C$ is continuous when f is continuous at all noints of D i.e.. for a given $\varepsilon > 0$. there $\exists$ a $\delta > 0$ such that

$|f(z) - f(z_0)| < \varepsilon$ whenever $z \in D \& |z - z_0| < \delta$ or equivalently, $f(z) \in B(f(z_0); \varepsilon)$ whenever $z \in B(z_0; \delta) \cap D$.

Results on Continuity

1. A function is continuous by default at all isolated points of the domain.
2. Continuous function maps connected sets to connected set.
3. Continuous function maps compact set to compact set.
4. If a continuous function is defined on a compact set then it is uniformly continuous.
5. If the function $f(x) = u(x,y) + iv(x,y)$ is continuous at $z_0 = x_0 + iy_0$, then the real functions $u(x,y) \& v(x,y)$ are also continuous at the point (x_0, y_0).
6. If $f(z) \& g(z)$ are continuous at a point z_0, then the function $f(z) \pm g(z), f(z)\cdot g(z), |f|, \frac{f(z)}{g(z)}$ where $g(z_0) \neq 0$ are also continuous at z_0.
7. If $f(z)$ is continuous in a closed region S, then it is bounded i.e., $|f(z)| \leq M\ \forall z \in S$.
8. The function $f(z)$ is continuous at $z = 0$ if the function $f\left(\frac{1}{\xi}\right)$ is continuous at $\xi = 0$.

   Composite Function: Let a function $g(z)$ be defined in the neighborhood of a point z_0 & let the image of $g(z)$ in this neighborhood be contained in a region in which $f(z)$ is defined. Then the composite function $f(g(z))$ is defined for all z in the neighborhood of the point z_0.
9. The composition of two continuous functions is continuous i.e., if $f: D_1 \to D_2$ is continuous at $z_0 \in D_1$ & if $g: D_2 \to C$ is continuous at $w_0 = f(z_0)$, then gof defined by $g \circ f(z) = g(f(z))$ is continuous at z_0.

10. A function f is continuous at a point $z_0 \in D$ if and only if $f(z_0) = lim_{n\to\infty} f(z_n)$ for every sequence $\langle z_n \rangle$ such that $z_n \in D$ for $n = 1,2,\dots.\&z_n \to z_0$ as $n \to \infty$.
11. Let $f: D \to C$ be a function. Then f is continuous on D if and only if for every onen set $O \subset C$ $f^{-1}(O) = \{z \in D: f(z) \in O\}$ is onen in D.

Examples:

(i) The functions $e^z, sin\ z, cos\ z$ are continuous function for all z.

Consider $e^z = e^{x+iy}$

$$= e^x cos\,y + ie^x sin\,y = u + iv$$

$$sin\,z = sin\,x\,cos\,h\,y + ico\,s\,x\,sin\,h\,y = u + iv$$

$$cos\ z = cos\ xcosh\ y - isin\ xsinh\ y = u + iv$$

since the real valued functions $u\&v$ of two real variables $x\&y$ are continuous $\forall x\&y$ in each case, the given function of a complex variable z are continuous for all z.

(ii) Show that the function $f(z)$ is not continuous at $z = 0$, where

$$f(z) = \{\frac{Imz}{|z|}, z \neq 0\ 0, z = 0$$

Solution: $\lim_{z\to 0} \frac{Im(z)}{|z|} = \lim_{x\to 0\ y\to 0} \frac{y}{\sqrt{x^2+y^2}}$

Consider the path $y = mx$. Then

$$\lim_{x\to 0\ y\to 0} \frac{mx}{\sqrt{x^2 + m^2x^2}} = \frac{m}{\sqrt{1 + m^2}}$$

Since the limit of $f(z)$ is path dependent so function is not continuous.

Uniform Continuity

A function $f(z)$ is said to be uniformly continuous in a region S if for a given real number $\varepsilon > 0$ depending only on ε such that

$$|f(z_1) - f(z_2)| < \varepsilon\ whenever\ 0 < |z_1 - z_2| < \delta$$

where $z_1\&z_2$ are any two points in the region S.

Note: It is meaningless to talk about uniform continuity at a point.

(i) If $f(z)$ is uniformly continuous in a region S, then $f(z)$ is continuous in S.

Example: Show that the function $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$.

Solution: We need to show that for any given $\varepsilon > 0, \exists$ a $\delta > 0$ such that $|z_1^2 - z_2^2| < \varepsilon$ whenever $0 < |z_1 - z_2| < \delta$ where $z_1\&z_2$ are any two points in the region $|z| < 1$, we have

$$|z_2^2 - z_1^2| = |z_2 - z_1||z_2 + z_1| \leq (|z_2| + |z_1|)(|z_2 - z_1|) \leq 2|z_2 - z_1|$$

Thus when $|z_1^2 - z_2^2| < \varepsilon$ it follows that $|z_2 - z_1| < \frac{\varepsilon}{2}$.

Thus, $\delta < \frac{\varepsilon}{2}$ depends only on $\varepsilon\&$ not on the choice of the points z_1, z_2 in the region. Hence $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$.

Differentiability

A complex function f defined on a non-empty set D is differentiable at $z_0 \in D$ if the limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \#(i)$$

exists & the value of the limit denoted by $f'(z_0)$ is called the derivative of f at z_0. The function f is differentiable on D if it is differentiable at every point of D or in terms of $\varepsilon - \delta$ notation, the limit in-equation (i) exists if and only if given $\varepsilon > 0, \exists$ a $\delta = \delta(\varepsilon, z_0) > 0$ such that

$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

Examples:

(i) $p(z) = \sum_{i=1}^{n} a_i z^i = f_1(z)$

(ii) $sin\ z = f_2(z)$

(iii) $cos\ z = f_3(z)$

(iv) $e^z = f_4(z)$

(v) $\alpha f_i(z) + \beta f_j(z) = f_5(z), j = 1$ to 4

(vi) $\frac{f_i(z)}{f_4(z)}, i = 1$ to 5 .

The functions defined above are all differentiable on the complex plane

Note: Differentiability ⇒ Continuity

Cauchy-Riemann Equations

Let $f: G \to C$ be differentiable & let $u(x,y) = Ref(z), v(x,y) = Imf(z)$ for $z \in G$. Let us evaluate the limit $f'(z) = \lim\limits_{h\to 0} \frac{f(z+h)-f(z)}{h}$ in two different ways. First let $h \to 0$ through real values of h for $h \neq 0, h$ real we get

$$\frac{f(z+h)-f(z)}{h} = \frac{f(x+h+iy)-f(x+iy)}{h}$$

$$= \frac{u(x+h,y)-u(x,y)}{L} + i\frac{v(x+h,y)-v(x,y)}{L}$$

Letting $h \to 0$ gives

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \ldots \text{(i)}$$

Now let $h \to 0$ through purely imaginary values; i.e., for $h \neq 0$ and h real,

$$\frac{f(z+ih)-f(z)}{ih} = -i\frac{(u(x,y+h)-u(x,y)}{h} + \frac{v(x,y+h)-v(x,y)}{h}$$

Thus $f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ … (ii)

equating the real & imaginary parts of (i) & (ii) we get the $C - R$ equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Polar Form of $C - R$ Equation

Let $f(z) = u + iv$ is a differentiable function &$z = re^{i\theta}$ then $C - R$ equations are given by

$$u_r = \frac{v_\theta}{r} \ \& \ v_r = -\frac{u_\theta}{r}$$

Complex Form of $C - R$ Equation

Let $f(z) = u + iv$ be a differential function.

Let $z = x + iy = z(x, y)$

& $z^{-} = x - iy = z^{-}(x, y)$

$x = \frac{z + z^{-}}{2}, y = \frac{z - z^{-}}{2i}$

i.e. $x = x(z, z^{-}), y = y(z, z^{-})$

$\Rightarrow \frac{\partial f}{\partial z^{-}} = \frac{\partial u}{\partial z^{-}} + i\frac{\partial v}{\partial z^{-}}$

$= \frac{\partial u}{\partial x}\frac{\partial x}{\partial z^{-}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z^{-}} + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial z^{-}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial z^{-}}\right)$

$= \frac{1}{2}u_x + \frac{1}{2}u_y + i\left(\frac{1}{2}v_x + \frac{1}{2}v_y\right)$

$= \frac{1}{2}\left[\left(u_x - v_y\right) + i\left(u_y + v_x\right)\right]$

Since f is differentiable, so it must satisfy the $C - R$ equation i.e.,

$$u_x = v_y \& u_y = -v_x$$

$\Rightarrow \frac{\partial f}{\partial z^{-}} = 0$ is called $C - R$ equation in complex form.

Necessary Condition for Differentiability

Let $f(z) = u + iv$ be a function then necessary condition f_0 it be differentiable is that u_x, u_y, v_x & v_y exist & satisfy $u_x = v_y \& u_y = -v_x$ i.e., $C - R$ equation

i.e. $\frac{\partial f}{\partial z^{-}} = 0$

Sufficient Condition for Differentiability

Let $f: S \to C$ where $S \subset C$ be such that $f(z) = u + iv$ & $(x_0 + iy_0) = z_0 \in S \cap S'$ then f is differentiable if

(i) u_x, u_y, v_x, v_y exist

(ii) All the partial derivative are continuous.

(iii) Also satisfy the $C - R$ equation.

Theorem: A real valued function of a complex variable either has derivative zero or the derivative does not exist.

Proof: Suppose that $f(z)$ is a real-valued function of complex variable whose derivative exists at a point a. Then

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

If we take the limit $h \to 0$ along the real axis, then $f'(a)$ is real. If we take the limit $h \to 0$ along the imaginary axis, then $f'(a)$ becomes a purely imaginary number. We must have $f'(a) = 0$.

Theorem: If f and g are differentiable at z_0, then their sum $f + g$, difference $f - g$, product fg, quotient f/g(where $g(z_0) \neq 0$) and the scalar multiplication cf, are also differentiable at z_0 and

$$(f + g)' = f' + g', (fg)' = f'g + fg',$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, (cf)' = cf'$$

Where c is a complex constant.

More generally, a finite linear combinations (of the form $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n, \alpha_j \in C, j = 1,2,\ldots,n$) and finite products of functions differentiable at z_0 are also differentiable at z_0.

Example: Let $f(z) = Rez$ and z_0 be an arbitrary fixed point in C. The $h = h_1 + ih_2 (\neq 0)$,

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{\text{Re}(h)}{h} = \begin{cases} 1 & \text{for } h = h_1 + i.0 \in \text{R} \setminus \{0\} \\ 0 & \text{for } h = 0 + ih_2 \in i(\text{R} \setminus \{0\}) \end{cases}$$

So that $lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ does not exist at z_0.

does not exist. Therefore, $f(z) = Rez$ is nowhere differentiable even though it is continuous in C.

Similarly, we see that $Imz, |z|, z^{-}$ and $Argz$ are all nowhere differentiable in C.

Is each of the functions listed here continuous in C ? Is each of these functions infinitely often real differentiable in R^2 ? Is a product of two nowhere differentiable functions always nowhere differentiable? **How about $g(z) = (z^{-})^2$?**

Example: Each of the functions

$f_1(z) = |z|, f_2(z) = Rez$ and $f_3(z) = Imz,$

is a non-constant real-valued function defined in C. Each of them is nowhere analytic. If we rewrite these function as

$f_1(x,y) = \sqrt{x^2+y^2}, f_2(x,y) = x, f_3(x,y) = y,$

then, except f_1, each of these functions are real differentiable in R^2.

Example: It is easy to see that the function f defined by

$$f(z) = |RezImz|^{1/2}$$

satisfies the $C - R$ equations at the origin, but is not differentiable at this point. To see this, we may rewrite the given function as

$f(z) = \frac{|z^2 - z^{-2}|^{1/2}}{2}$ or $f = u + iv$ with $u(x,y) = |xy|^{1/2}$ and $v(x,y) = 0$.

Note that f is identically zero on the real and imaginary axes. Therefore, it is trivial to see that $u_x(0,0) = u_y(0,0) = v_x(0,0) = v_y(0,0) = 0$. For example,

$u_x(0,0) = lim_{s\to 0} \frac{u(s,0)-u(0,0)}{s} = 0.$

Thus, the $C - R$ equations hold at $z = 0$. However, taking $h = re^{i\theta} \neq 0$ with $r \to 0$, we find that

$$lim_{h\to 0} \frac{f(h) - f(0)}{h} = lim_{r\to 0} \frac{|r^2 cos\ \theta sin\ \theta|^{1/2}}{r(cos\ \theta + isin\ \theta)} = \frac{e^{-i\theta}|sin\ 2\theta|^{1/2}}{\sqrt{2}}$$

Which is clearly depending upon θ (e.g. take $\theta = 0$ and $\theta = \pi/4$). We conclude that f is not differentiable at $z = 0$ even though f satisfies the $C - R$ equations at the origin. Here, since $v(x,y) = 0, v$ is a C^{∞}-function in R^2. Are the partial derivatives u_x and u_y continuous at the origin? How about the functions

$f(z) = |RezImz|^{1/3}$ and $f(z) = |RezImz|^{1/4}$?

Example: The function $f(z) = |z|$ is now here differentiable

Since $f(z) = |z| = \sqrt{x^2+y^2} = \sqrt{z \cdot z^{-}} \Rightarrow \frac{\partial f}{\partial z^{-}} = \sqrt{z} \times \frac{1}{2\sqrt{z}} \neq 0$

$\Rightarrow f(z) = |z|$ does not satisfy the $C - R$ equation hence it is nowhere differentiable.

Example: The function $f(z) = |z|^2$ is differentiable only at origin.

Since $f(z) = |z|^2 = x^2 + y^2 = z \cdot z^-$

$\Rightarrow \frac{\partial f}{\partial z^-} = z$, so if $f(z)$ wants to be differentiable then it must satisfy the

$C - R$ equation i.e., $\frac{\partial f}{\partial z^-} = 0 \Rightarrow z = 0$

Consider $lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = lim_{z\to 0} \frac{z\cdot z^-}{z} = lim_{z\to 0} z^- = 0$

Hence $f(z)$ is differentiable only at origin.

Example: Function $f(z) = \frac{1}{z}$ is differentiable at all points except at $z = 0$.

Singularities Of Analytic Functions

Regular Point

Let f is defined in D and z_0 is an interior point in D. We say f is regular on z_0, or equivalently, z_0 is a regular point of f if $\exists \delta > 0$ such that $f'(z)$ exists $\forall z \in |z - z_0| < \delta$

Statement: If f is regular at z_0, then $\exists$ a open disc around z_0 such that every point of the disc is regular point.

$f'(z)$ Exists $\forall z \in |z - z_0| < \delta$

Let $z_1 \in |z - z_0| < \delta$ and $|z_0 - z_1| = \delta_1, \delta_2 = \delta - \delta_1$

$$\delta' = min\{\delta_1, \delta_2\}$$

$$|z - z_1| < \delta' \subset |z - z_0| < \delta$$

$\Rightarrow z_1$ is regular point of f

Analytic Function

f is said to be analytic in D if it is regular at every point of D.

Notes:

(i) If f is analytic in $D \Rightarrow D$ is open

(ii) If it is said that f is analytic at z_0 it means f is analytic in an open disc around z_0. i.e. z_0 is a regular point.

(iii) If f is defined on D where D is open and connected then f is analytic on D iff f is differentiable at every point of D.

(iv) If f is analytic on $D - \{a\}$ where $a \in D$ then we say f is not analytic on a, if $lim_{z\to a} f(z)$ either doesn't exist if exist then not equal to $f(a)$.

Singularity

Let f is defined on D. Then a limit point of D i.e. the limit point of the set of regular point is defined as singular point if it is not regular itself.

If $D =$ Domain of definition

$R =$ The set of regular points

$R' =$ The set of limit point of R

Then if $\alpha \in R' - R$

$\Rightarrow \alpha$ is a limit point of R but $\alpha \notin R$ then α is a singular point.

Example: Let $f(z) = log\ z$

then $D = \{z \in C - \{x + iy: x \leq 0, y = 0\}\}$

$D = C - \{x + iy: x \leq 0, y = 0\}$

$R = D$

$R' = C$
$S = \{x + iy : x \leq 0, y = 0\}$

Result: Set of all singular points is a closed set i.e. $S' = S$.
Proof: Define S' = The set of limit point of S
If $\alpha \in S' \Rightarrow \alpha \in R$ … (i)
$\Rightarrow |z - \alpha| < \delta \cap S$ is infinite for some $\delta > 0$
$\Rightarrow |z - \alpha| < \delta \cap R'$ is infinite
$\Rightarrow |z - \alpha| < \delta \cap R$ is infinite
$\Rightarrow \alpha \in R'$
$\Rightarrow \alpha \in S \Rightarrow S' \subseteq S$.

Classification of Singular Points

Positional Classification :

I. Isolated Singularity: Let $z_0 \in S$. Then a singular point is called an isolated singularity if $\exists \delta > 0$ such that f is regular in $0 < |z - z_0| < \delta$ i.e., $I = S = S'$ = The set of isolated points in S.

Examples: Let $tan\ \frac{1}{z} = \frac{sin\frac{1}{z}}{cos\frac{1}{z}}$, then $f(z)$ have singularities at those points where $cos\ \frac{1}{z} = 0$ and at $z = 0$.

i.e. $S = \left\{\frac{1}{(2n+1)\pi/2}, n \in Z\right\} \cup \{0\}$
$S' = \{0\}$
$\Rightarrow$ 0 is not an isolated singularity.

II. Non-isolated Singularity: The limit point of singularity is defined as non-isolated singularity.
So, α is non-isolated singularity iff $\alpha \in S'$.
i.e., α is non-isolated singularity iff it is not isolated singularity

Examples:

(i) $f(z) = log\ z$
Every singularity of $log\ z$ is non-isolated singularity

(ii) $f(z) = \frac{1}{sin\,(\pi/z)}, S = \left\{\frac{1}{n}, n \in Z - \{0\}\right\} \cup \{0\}$
$S' = \{0\}$
$\Rightarrow \{0\}$ is non-isolated singularity, rest are isolated singularity.

(iii) $f(z) = \frac{1}{sin\ \pi z}$
$S = Z, S' = \phi$

Notes:
(i) $|z| > R$ is a neighborhood of ∞ for any $R > 0$.
(ii) In extended C, ∞ is limit point of every unbounded set.

Examples: In extended $C, Z' = \{\infty\}$

$$N' = \{\infty\}$$

(iii) The behavior of f at ∞ is analyzed with help of that $g(z)$ at $z = 0$ where $g(z) = f\left(\frac{1}{z}\right)$

Example: Let $f(z) = sin\ z$
Then $f\left(\frac{1}{z}\right) = sin\ \frac{1}{z}$ has isolated singularity at $z = 0$

$\Rightarrow f(z)$ has isolated singularity at ∞.

(iv) If g is regular at 0 then f is regular at ∞ where $f\left(\frac{1}{z}\right) = g(z)$

(v) If set of singular point of f is unbounded then ∞ is non-isolated singularity of f.

Example: $f(z) = \frac{1}{\sin z - \cos z}$

It has infinite singularity which are at the points

$n\pi + \frac{\pi}{4}, n \in Z$ and ∞ is non isolated singularity.

Character Based Classification of Singularity

1. **Removable Singularity:** $z_0 \in S$ is said to be removable singularity. If $\lim_{z\to z_0} f(z)$ exists finitely, where f is defined in $0 < |z - z_0| < \delta$ for some $\delta > 0$

   Examples:

   (i) $f(z) = \sin z \, z \neq 0$

   (ii) $f(z) = \frac{\sin z}{z}$

   (iii) $f(z) = \{\frac{\sin z}{z}, z \neq 0 \; 0, z = 0$

   $$f(z) = z, |z| < 1 \;\; D = |z| < 1 \;\; R = D, R' = |z| \leq 1 \;\; S = |z| = 1$$

   Note: Limit of Branch point is not defined.

   Pole: Let $z_0 \in S$ and $\lim_{z\to z_0} f(z)$ does not exist, then z_0 is called a pole of order m. if $\lim_{z\to z_0} (z - z_0)^m f(z)$ exists non-zero.

   Example: $f(z) = \frac{1}{z}$ – order 1 pole

   $$f(z) = \frac{\sin z}{z^3} - \text{ order } 2 \text{ pole} \quad f(z) = \frac{\cos z}{z^3} - \text{ pole order } 3$$

   Order one pole is called simple pole.

2. **Essential Singularity:** Which is neither the above two is called essential singularity.

   Example: $\log z$ has essential singularity at the points on the negative x axis.

   Example: $f(z) = \{1 \; |z| < 1 \; 2 \; 1 \leq |z| < 2$

   $$D = |z| = 2$$

   So, $|z| = 1$ is essential singularity and non-isolated.

   $$|(z - z_0)^m \cdot f(z)| \leq 2|z - z_0|^m < \varepsilon$$

   As $D = |z| < 2$

   $$R = D - \{|z| = 1\} \;\; R' = |z| \leq 2$$

   Result: If $f(z)$ is defined in a neighborhood of z_0 except possibly at z_0.

   Case (i): When z_0 is regular point. Then

   (a) $f(z_0), f'(z_0)$ exist.

   (b) $f'(z)$ exist $\forall z \in |z - z_0| < \delta$ for some $\delta < 0$

   (c) f is bounded in a neighborhood of z_0

   Case (ii): If z_0 is removable singularity

   (a) By definition it is an isolated singularity.

   (b) As limit exist in a neighborhood of z_0 function is bounded.

Case (iii): If z_0 is pole

(a) By definition it is an isolated singularity.

(b) $\lim_{z \to z_0} f(z) = \infty$ i.e., in a neighborhood of pole functions are unbounded

(c) If $z = z_0$ is a pole of order n of $f(z)$ then $(z - z_0)^n f(z)$ has a removable singularity at $z = z_0$ i.e. $\lim_{z \to z_0} (z - z_0)^n f(z)$ exists and non zero.

Result: If $f \& g$ are two analytic function defined on some domain D such that f has a zero of order $n \& g$ has a zero of order m at $z = z_0$ then

(i) $f(z) = (z - z_0)^n f_1(z) \& g(z) = (z - z_0)^m g_1(z)$ where

$$f_1(z_0) \neq 0 \& g_1(z_0) \neq 0$$

(ii) Define $h(z) = \frac{f(z)}{g(z)}$ then

(a) $z = z_0$ is a removable singularity if $n \geq m$

(b) If $n < m$ then at $z = z_0 h(z)$ has a pole of order $m - n$.

Consider $\frac{\pi}{z_n} = 2n\pi + \frac{\pi}{6}$

$\Rightarrow z_n = \frac{1}{2n + \frac{1}{6}} \to 0 \;\; f(z_n) \to \frac{1}{2} (\, Essential \,)$

Case (iv): When z_0 is isolated essential singularity.

f is unbounded in a neighborhood of z_0. In this case

$$A = \{ lim f(z)\; can\; vary : z_n \to z_0 \} \equiv C$$

Entire Function

f is said to be entire if it is analytic on C i.e. regular on C

i.e. $R = C$

i.e. ∞ is the only possible singularity.

If f is entire then $e^{f(z)}$ is entire. In fact singularities of f and ef are same nacitionallv not character wise.

If z_0 is pole or essential of $f \Rightarrow z_0$ is essential of e^f.

Example: Let $f(z) = \frac{1}{z}$ then $\frac{1}{z}$ has a pole of order 1 .

$\Rightarrow e^{1/z}$ has essential singularity at $z = 0$

Results on Analyticity

Let f be an analytic function defined on a domain D (open and connected) then

1. $f'(z) = 0 \forall z \in D$ then f is constant.

2. $f(z) = u + iv$, If any of u and v is constant then f is constant.
   Let $u(x, y) = k$
   $u_x = 0 \& u_y = 0$
   Since $u_x = v_y = 0$ & $u_y = -v_x = 0$
   $\Rightarrow f'(z) = 0 \Rightarrow f(z) = k, k \in C.$

3. If $|f|$ is constant then f is constant.
   $|f| = c \Rightarrow u^2 + v^2 = c$
   If $c = 0$ then done
   If $c \neq 0 \Rightarrow 2uu_x + 2vv_x = 0$

& $uu_y + vv_y = 0$

i.e. $vv_x + uv_y = 0$

$-uv_x + vv_y = 0$

$\Rightarrow [v\ u\ \ -u\ v\ ][v_x\ v_y\ ] = [0\ 0\ ]$

$Ab = 0$ where $A = [v\ u\ \ -u\ v\ ]$&$b = [v_x\ v_y\ ]$

Since $|A| = u^2 + v^2 \neq 0 \Rightarrow$ system of equation has unique solution $\Rightarrow v_x = v_y = 0$ i.e. $v(x, y)$ is a constant function and hence f is constant.

Case (iv): If $arg\ f =$ constant then f is constant.

Let $arg\ f = tan^{-1}\frac{v}{u} = k \Rightarrow \frac{v}{u} = tan\ k = \lambda$

$\lambda u - v = 0$

$\Rightarrow \lambda u_x - v_x = 0 \Rightarrow \lambda u_x + u_y = 0$

&$\lambda u_y - v_y = 0 \Rightarrow -u_x + \lambda u_y = 0$

$\Rightarrow [\lambda\ 1\ \ -1\ \lambda\ ][u_x\ u_y\ ] = [0\ 0\ ]$ Which is system of equation and have trivial **solution i.e.** $u(x, y)$ constant and hence function $f(z)$ is constant.

4. If $au + bv =$ constant then function is constant.

   Let $f(z) = u + iv$

   & $au + bv = c$

   $g(z) = (a + ib)f(z) = (a - ib)(u + iv)$

   $= \ (au + bv + i(av - bu))$

   $= \ U + iV,\ since\ U = au + bv\ is\ constant$

   $\Rightarrow g(z)$ is constant and hence $f(z)$ is constant.
5. If u and v lie on circle then function is constant.
6. If u and v are harmonic conjugate of each other then f is constant i.e. $f = u + iv$ is constant.
7. If $f = u + iv$ is analytic then u and v are harmonic functions of x and y

   i.e., u and v satisfies Laplacian equation

   i.e., $\Delta^2 u = 0$ i.e. $u_{xx} + u_{yy} = 0$

   $\Delta^2 v = 0$ i.e. $v_{xx} + v_{yy} = 0$

   Example: Let $f(z) = x^2 + y^2 + 2xyi = u + iv$

   Since u is not harmonic $\Rightarrow$ Not analytic

   $$f(z) = 2xy + (x^2 - y^2)i = u + iv$$

   Then $u_{xx} + u_{yy} = 0$ but f is not analytic
8. If $f(z) = u + iv$ is analytic then $u(x, y) = \alpha, v(x, y) = \beta$ represents orthogonal family of curves with α and β aress parameter.

Construction of Analytic Function

Method 1: Milne's Thomson's method.

We have $z = x + iy$ so that $x = \frac{z+z^-}{2}, y = \frac{z-z^-}{2i}$.

$w = f(z) = u + iv = u(x, y) + iv(x, y)$

or $f(z) = u\left(\frac{z+z^-}{2}, \frac{z-z^-}{2i}\right) + iv\left(\frac{z+z^-}{2}, \frac{z-z^-}{2i}\right)$

In fact, this relation is formal identity in two independent variables z and z^-