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SIERRA INNOVATIONS

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1 CHAPTER Integral equation

Integral euqation

Definition

An integral equation is an equation in which an unknown function, to be determined, appears under one or more integral signs. If the derivatives of the function are involved, it is called an integro-differential equation.

An equation of the form

$$\alpha(x)\phi(x) + F(x) + \lambda \int_{\Omega} K(x,\xi)\phi(\xi)d\xi = 0,$$

is called the linear integral equation, where $\phi(x)$ is the unknown function; $\alpha(x), F(x)$ and the kernel of the integral equation $K(x,\xi)$ are known functions; λ is a non-zero real or complex parameter, and the integration extends over the domain Ω of the auxiliary variable ξ.

Linear integral equations are classified into two used types:

Volterra Integral Equation

An integral equation is said to be a Volterra-integral equation if the upper limit of integration is a variable, **e.g.,**

$$\alpha(x)\phi(x) = F(x) + \lambda \int_{a}^{x} K(x,\xi)\phi(\xi)d\xi$$

(i) When $a = 0$, the unknown function ϕ appears only under the integral sign and nowhere else in the equation, then

$$F(x) = \lambda \int_{0}^{x} K\,(x,\xi)\phi(\xi)d\xi, a > -\infty$$

is called the Volterra's integral equation of first kind.

(ii) When $a = 1$, the equation involves the unknown function ϕ, both inside as well as outside the integral sign, then

$$\phi(x) = F(x) + \lambda \int_{1}^{x} K\,(x,\xi)\phi(\xi)d\xi$$

is called the Volterra's equation of second kind.

(iii)When $a = 1, F(x) = 0$, the equation reduced to

$$\phi(x) = \lambda \int_{1}^{x} K\,(x,\xi)\phi(\xi)d\xi$$

is called the homogeneous Volterra's integral equation of second kind.

Fredholm Integral Equations

An integral equation is said to be Fredholm integral equation if the domain of integration Ω is fixed, e.g.,

$$\alpha(x)\phi(x) = F(x) + \lambda \int_{a}^{b} K\,(x,\xi)\phi(\xi)d\xi$$

(i) When $a = 0$, the equation involves the unknown function † only under the integral sign, then

$$F(x) = \lambda \int_a^b K(x, \xi)\phi(\xi)d\xi, a \le x \le b$$

is called the Fredholm integral equation of first kind.

(ii) When $a = 1$, the equation involves the unknown function ϕ both inside as well as outside the Integral sign, then

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\phi(\xi)d\xi, a \le x \le b$$

is called the non-homogeneous Fredholm integral equation of second kind.

(iii)When $a = 1, F(x) = 0$, the equation reduced to

$$\phi(x) = \lambda \int_a^b K(x, \xi)\phi(\xi)d\xi, a \le x \le b$$

is called as the homogeneous Fredholm integral equation of second kind.

Differentiation of Function Under an Integral Sign

Consider the function $I_a(x)$ defined by the relation

$$I_a(x) = \int_a^x (x-\eta)^{n-1} f(\eta)d\eta \quad \ldots (1)$$

Where n is a positive integral and a is a constant.
We known that

$$\frac{d}{dx}\int_{P(x)}^{Q(x)} F(x,\eta)d\eta = \int_P^Q \frac{\partial}{\partial x}\{F(x,\eta)\}d\eta + F[x, Q(x)]\frac{dQ}{dx} - F[x, P(x)]\frac{dP}{dx}$$

Which is valid if F and $\frac{\partial F}{\partial x}$ are continuous functions of both x, η and th first derivative of $P(x)$ and $Q(x)$ are continuous.
Differentiating (1) under the integral sign, we have

$$\frac{dI_n}{dx} = (n-1)\int_a^x (x-\eta)^{n-2} f(\eta)d\eta \times [(x-\eta)^{n-1} f(\eta)]_{\eta=x}$$

$$\frac{dI_n}{dx} = (n-1)I_{n-1}, n > 1 \quad \ldots (2)$$

From the relation (1), we have

$$I_1(x) = \int_a^x f(\eta)d\eta \Rightarrow \frac{dI_1}{dx} = f(x) \quad \ldots (3)$$

Differentiating (2) successively m times, we have

$$\frac{d^m I_n}{dx^m} = (n-1)(n-2)(n-3)\ldots(n-m)I_{n-m}, n > m$$

In particular, we have

$$\frac{d^{n-1} I_n}{dx^{n-1}} = (n-1)!\, I_1(x)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{d^{n-1} I_n}{dx^{n-1}}\right) = (n-1)!\frac{dI_1}{dx} = (n-1)!\, f(x) \quad \ldots (4)$$

Thus, we have

$$I_1(x) = \int_a^x f(x_1)dx_1 \quad \frac{dI_2}{dx} = I_1 = \int_a^x f(x_1)dx_1 \quad \Rightarrow I_2(x) = \int_a^x \int_a^{x_2} f(x_1)dx_1 dx_2$$

In general, we have

$$I_n(x) = (n-1)! \int_a^x \int_a^{x_n} \ldots \ldots \int_a^{x_3}\int_a^{x_2} f(x_1)dx_1 dx_2 \ldots\ldots dx_{n-1}dx_n \quad \ldots (5)$$

From the relations (1) and (5), we have

$$\int_a^x \int_a^{x_n} \ldots.. \int_a^{x_3} \int_a^{x_2} f\ (x_1)dx_1dx_2 \ldots\ldots dx_{n-1}dx_n \quad = \frac{1}{(n-1)!} I_n(x)$$

$$= \frac{1}{(n-1)!} \int_a^x (x-\eta)^{n-1} f(\eta)d\eta$$

This may be represented as the result of integrating the function f from a to x and then integrating $(n-1)$ times, we have

$$\int_a^x f\ (\eta)d\eta^n = \int_a^x \frac{(x-\eta)^{n-1}}{(n-1)!} f(\eta)d\eta \qquad \ldots (6)$$

Relation between Differential and Integral Equations

There is a fundamental relationship between integral equations and ordinary and partial differential equations with given initial values. Consider the differential equation of nth order as

$$\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots. + a_n(x)y = F(x) \qquad \ldots (1)$$

with continuous coefficients $a_i(x) = i = 1,2,3, \ldots, n$. The initial conditions are prescribed as follows:

$$y(0) = C_0, y'(0) = C_1, y''(0) = C_2, \ldots, y^{n-1}(0) = C_{n-1} \qquad \ldots (2)$$

Where the prime denotes differentiation with respect to x. Consider $\frac{d^ny}{dx^n} = \phi(x)$ By integrating and using the initial conditions (2), we have

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_0^x \phi\ (\xi)d\xi + C_{n-1} \quad \frac{d^{n-2}y}{dx^{n-2}} = \int_0^x \phi\ (\xi)d\xi^2 + C_{n-1}x + C_{n-2}$$

$$y = \int_0^x \phi\ (\xi)d\xi^n + C_{n-1}\frac{x^{n-1}}{(n-1)!} + C_{n-2}\frac{x^{n-2}}{(n-2)!} + \cdots. + C_0 \qquad \ldots (3)$$

Where $\int_0^x \phi\ (\xi)d\xi^n$ represents for a multiple integral of order n.

From the relations (3) and (1), we obtain

$$\phi(x) + a_1(x)\int_0^x \phi\ (\xi)d\xi + a_2\int_0^x \phi\ (\xi)d\xi^2 + \cdots.. + a_n(x)\int_0^x \phi\ (\xi)d\xi^n$$

$$= F(x) + \sum_1^n iC_i\chi_i(x) \qquad \ldots (4)$$

Where $\chi_i(x) = a_i(x) + \frac{x}{1!}a_{i+1}(x) + \cdots. + a_n(x)\frac{x^{n-i}}{(n-i)!}$ $\qquad \ldots (5)$

$$\phi(x) + \int_0^x \left[a_1(x) + a_2(x)(x-\xi) + \cdots \ldots + a_n(x)\frac{(x-\xi)^{n-1}}{(n-1)!}\right]\phi(\xi)d\xi = G_1(x), \qquad \ldots (6)$$

Where $G_1(x) = F(x) + \sum_1^n iC_i\chi_i(x)$

The equation (6) represents the non-homogeneous Volterra's integral equation of second kind.

Particular Case

Consider the linear differential equation of second order.

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x) \qquad \ldots (1)$$

with initial conditions

$$y(0) = C_0 \ and \ y'(0) = C_1 \qquad \ldots (2)$$

Consider $\frac{d^2y}{dx^2} = \phi(x)$

By integrating and using the initial conditions (2), we have

$$\frac{dy}{dx} = \int_0^x \phi(\xi) d\xi + C_1 \quad \ldots (3)$$

$$and\ y = \int_0^x (x-\xi)\, \phi(\xi) d\xi + C_1 x + C_0 \quad \ldots (4)$$

The given differential equation reduces to

$$\phi(x) + a_1(x)\left[\int_0^x \phi(\xi)d\xi + C_1\right] + a_2(x)\left[\int_0^x (x-\xi)\phi(\xi)d\xi + C_1 x + C_0\right] = F(x)$$

$$\text{or } \phi(x) + \int_0^x [a_1(x) + a_2(x)(x-\xi)]\phi(\xi)d\xi$$

$$= F(x) - C_1 a_1(x) - C_1 x a_2(x) - C_0 a_2(x)$$

$$\text{or } \phi(x) = f(x) + \lambda \int_0^\pi K(x,\xi)\phi(\xi)d\xi \quad \ldots (5)$$

Where $K(x,\xi) = a_1(x) + a_2(x)(x-\xi), \lambda = -1$

$$f(x) = F(x) - C_1 a_1(x) - C_1 x a_2(x) - C_0 a_2(x) \quad \ldots (6)$$

Which represents the Volterra's integral equation of the second kind. Similarly, the boundary value problems in ordinary differential equations lead to Fredholm integral equations.

Example: Show that the function $\phi(x) = (1+x^2)^{-3/2}$ is a solution of the Volterra integral equation

$$\phi(x) = \frac{1}{1+x^2} - \int_0^x \frac{\xi}{1+x^2}\phi(\xi)d\xi$$

Solution: Substituting the function $\phi(x) = (1+x^2)^{-3/2}$ in the given equation, we have

$$\frac{1}{1+x^2} - \int_0^x \frac{\xi}{1+x^2} \cdot \frac{1}{(1+\xi^2)^{3/2}} d\xi$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2}\left\{\frac{1}{(1+\xi^2)^{1/2}}\right\}_0^x$$

$$= \frac{1}{1+x^2} + \frac{1}{(1+x^2)^{3/2}} = \frac{1}{1+x^2} = \frac{1}{(1+x^2)^{3/2}} = \phi(x)$$

The substitution of $\phi(x)$ reduces the given equation to an identity with respect to x, thus $\phi(x) = (1+x^2)^{-3/2}$ is a solution of the integral equation.

Example: From an integral equation corresponding to the differential equation

$$\frac{d^2y}{dx^2} - \sin\ x \frac{dy}{dx} + e^x y = x$$

with the initial conditions

$$y(0) = 1, y'(0) = -1\#(1)$$

Solution: Consider $\frac{d^2y}{dx^2} = \phi(x)$ … (1)

Then $\frac{dy}{dx} = \int_0^x \phi(\xi) d\xi = -1$ … (2)

And $y = \int_0^x (x-\xi)\phi(\xi) d\xi - x + 1$ … (3)

Substituting the relations (1), (2) and (3) in the given differential equation, we have

$$\phi(x) - \sin\ x\left[\int_0^x \phi(\xi)d\xi - 1\right] + e^2\left[\int_0^x (x-\xi)\phi(\xi)d\xi - x + 1\right] = x$$

$$\Rightarrow \phi(x) = [x - \sin\ x + e^x(x-1)] + \int_0^x [\sin\ x - e^x(x-\xi)]\phi(\xi)d\xi$$

represents a Volterra's integral equation of second kind.

Example: Reduce the initial integral equation of second kind.

$$\phi'(x) + \lambda\phi(x) = F(x),$$

$$\text{with } \phi(0) = 1, \phi'(0) = 0.$$

Solution: The differential equation is given as

$$\phi'(x) + \lambda\phi(x) = F(x)$$

$$\Rightarrow \phi'(x) = F(x) - \lambda\phi(x)$$

Integrating both the sides with regard to x, we have

$$\int_0^x \phi''(x)dx = \int_0^x \{F(x) - \lambda\phi(x)\}dx$$

$$\Rightarrow [\phi'(x)]_0^x = \int_0^x \{F(x) - \lambda\phi(x)\}dx$$

$$\Rightarrow \phi'(x) - \phi'(0) = \int_0^x \{F(x) - \lambda\phi(x)\}dx$$

$$\Rightarrow \phi'(x) = \int_0^x \{F(x) - \lambda\phi(x)\}dx$$

Integrating both the sides with regard to x, we have

$$\int_0^x \phi'(x)dx = \int_0^x \{\int_0^x \{F(x) - \lambda\phi(x)\}dx\}dx^2$$

$$\Rightarrow \phi(x) - \phi(0) = \int_0^x \{F(\xi) - \lambda\phi(\xi)\}d\xi^2$$

$\Rightarrow \phi(x) = 1 + \int_0^x \{(x - \xi)[F(\xi) - \lambda\phi(\xi)]\}d\xi,$

Which reduces to a Volterra's integral equation of second kind.

Example: Reduce the differential equation

$$\phi''(x) - 3\phi'(x) + 2\phi(x) = 4\sin\ x$$

with the conditions

$\phi(0) = 1, \phi'(0) = -2$

into a non-homogeneous Volterra's integral equation of second kind.

Conversely, derive the original differential equation with the initial conditions from the integral equation obtained.

Solution: The given differential equation may be written as

$$\phi''(x) = 4\sin\ x - 2\phi(x) + 3\phi'(x) \qquad \dots (1)$$

integrating with regard to x, both the sides we have

$$\int_0^x \phi''(x)dx = 4\int_0^x \sin\ xdx - 2\int_0^x \phi(x)dx + 3\int_0^x \phi'(x)dx$$

$$\Rightarrow [\phi'(x)]_0^x = -4(\cos\ x)_0^x - 2\int_0^x \phi\ (x)dx + 3[\phi(x)]_0^x$$

$$\Rightarrow \phi'(x) - \phi'(0) = -4(\cos\ x - 1) - 2\int_0^x \phi\ (x)dx + 3[\phi(x) - 1]$$

$$\Rightarrow \phi'(x) = -2 - 4(\cos\ x - 1) + 3[\phi(x) - 1] - 2\int_0^x \phi\ (x)dx$$

$$\Rightarrow \phi'(x) = -1 - 4\cos\ x + 3\phi(x) - 2\int_0^x \phi\ (x)dx \qquad \dots (1)$$

integrating with regard to x, both the sides, we have

$$\int_0^x \phi'(x)dx = -\int_0^x dx - 4\int_0^x \cos xdx + 3\int_0^x \phi(x)dx - 2\int_0^x \phi(x)dx^2$$

$$\Rightarrow [\phi(x)]_0^x = -x - 4\sin\ x + 3\int_0^x \phi(x)dx - 2\int_0^x \phi(x)dx^2$$

$$\Rightarrow \phi(x) - \phi(0) = -x - 4\sin x + 3\int_0^x \phi(\xi)d\xi - 2\int_0^x \phi(\xi)d\xi^2$$

$$\Rightarrow \phi(x) - 1 = -x - 4\sin x + 3\int_0^x \phi(\xi)d\xi - 2\int_0^x (x-\xi)\phi(\xi)d\xi$$

$$\Rightarrow \phi(x) = (1 - x - 4sin\ x) + \int_0^x [3 - 2(x-\xi)]\ \phi(\xi)d\xi \qquad \ldots (3)$$

Represents the non-homogeneous Volterra's integral equation of second kind.

Converse: Again, differentiating the equation (3) with regard to x, we have

$$\phi'(x) = -1 - 4\cos x + \frac{d}{dx}\int_0^x [3 - 2(x-\xi)]\,\phi(\xi)d\xi$$

$$\Rightarrow \phi'(x) = -1 - 4\cos x + \int_0^x \frac{\partial}{\partial x}[3 - 2(x-\xi)]\phi(\xi)d\xi + [3 - 2(x-x)]\phi(x)\cdot 1$$

$$\Rightarrow \phi'(x) = -1 - 4cos\ x + 3\phi(x) - 2\int_0^x \phi(\xi)d\xi \qquad \ldots (4)$$

Differentiating both the sides with regard to x, we have

$$\Rightarrow \phi''(x) = 4sin\ x + 3\phi'(x) - 2\frac{d}{dx}\int_0^x \phi(\xi)d\xi$$

$$\Rightarrow \phi''(x) = 4sin\ x + 3\phi'(x) - 2\phi(x)$$

$$\Rightarrow \phi''(x) - 3\phi'(x) + 2\phi(x) = 4sin\ x$$

Which is the required given differential equation. Putting $x = 0$ in the equations (3) and (4), we have $\phi(0) = 1, \phi'(0) = -2$.

Example: Convert the differential equation $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$

With initial conditions $\phi(0) = 0, \phi'(0) = 0$, into Fredholm integral equation of second kind. Also, recover the original differential equation from the integral equation you obtain.

Solution: The differential equation may be written as

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \qquad \ldots (1)$$

Integrating both the sides with regard to x, we have

$$\int_0^x \frac{d^2\phi}{dx^2}dx = -\lambda\int_0^x \phi dx$$

$$\left(\frac{d\phi}{dx}\right)_0^x = -\lambda\int_0^x \phi dx$$

$$\phi'(x) - \phi'(0) = -\lambda\int_0^x \phi dx$$

Consider $\phi'(0) = C$, a constant, then $\phi'(x) = C - \lambda\int_0^x \phi(x)dx$

Integrating both the sides, we have

$$[\phi(x)]_0^x = Cx - \lambda\int_0^x \phi\,(x)dx^2$$

$$\Rightarrow \phi(x) - \phi(0) = Cx - \lambda \int_0^x (x-\xi)\phi(\xi)d\xi$$

$\Rightarrow \phi(x) = Cx - \lambda \int_0^x (x-\xi)\phi(\xi)d\xi$ ... (2)

Since $\phi(1) = 0 \Rightarrow \phi(1) = C(1) - \lambda \int_0^1 (1-\xi)\, \phi(\xi)d\xi$

$$\Rightarrow 0 = C - \lambda \int_0^1 (1-\xi)\phi(\xi)d\xi$$

$\Rightarrow C = \frac{\lambda}{1}\int_0^1 (1-\xi)\phi(\xi)d\xi$ ... (3)

From the relations (2) and (3), we have

$$\phi(x) = \frac{\lambda}{1} x \int_0^1 (1-\xi)\,\phi(\xi)d\xi - \lambda \int_0^x (x-\xi)\phi\,(\xi)d\xi$$

$$\Rightarrow \phi(x) = \int_0^1 \frac{\lambda x(1-\xi)}{1}\phi(\xi)d\xi + \int_0^x \frac{\lambda x(1-\xi)}{1}\phi(\xi)d\xi - \int_0^x \frac{\lambda(x-\xi)}{1}\phi(\xi)d\xi$$

$$\Rightarrow \phi(x) = \int_0^x \left[\frac{\lambda x(1-\xi)}{1} - \frac{\lambda(x-\xi)}{1}\right]\phi(\xi)d\xi + \int_x^1 \frac{\lambda x(1-\xi)}{1}\phi(\xi)d\xi$$

$$\Rightarrow \phi(x) = \lambda \int_0^x \frac{\xi\xi(1-\xi)}{1}\phi(\xi)d\xi + \int_x^1 \frac{x(1-\xi)}{1}\phi(\xi)d\xi$$

$\Rightarrow \phi(x) = \lambda \int_0^1 K\,(x,\xi)\phi(\xi)d\xi$ ... (4)

$K(x,\xi) = \begin{cases} \frac{\xi(1-\xi)}{1} & if\ 0 \le \xi \le x. \\ \frac{x(1-\xi)}{1} & if\ x \le \xi \le 1 \end{cases}$ ... (5)

This determines the homogeneous Fredholm integral equation of second kind.

Converse: The integral equation may be taken as

$\phi(x) = \int_0^l \frac{\lambda x(l-\xi)}{l}\phi(\xi)d\xi - \int_0^x \lambda\,(x-\xi)\phi(\xi)d\xi$ ... (6)

Differentiating both the sides with regard to x, we have

$$\phi'(x) = \frac{d}{dx}\int_0^l \frac{\lambda x(l-\xi)}{l}\phi(\xi)d\xi - \frac{d}{dx}\int_0^x \lambda\,(x-\xi)\phi(\xi)d\xi$$

$$\Rightarrow \phi'(x) = \int_0^l \frac{\partial}{\partial x}\left\{\frac{\lambda x(l-\xi)}{l}\phi(\xi)\right\}d\xi - \int_0^x \frac{\partial}{\partial x}\{\lambda(x-\xi)\}\phi(\xi)d\xi$$

$\Rightarrow \phi'(x) = \int_0^l \frac{\lambda(l-\xi)}{l}\phi(\xi)d\xi - \int_0^x \lambda\phi\,(\xi)d\xi$ ... (7)

again differentiating both sides with regard to x, we have

$$\Rightarrow \phi''(x) = \frac{d}{dx}\int_0^l \frac{\lambda(l-\xi)}{l}\phi(\xi)d\xi - \frac{d}{dx}\int_0^x \lambda\phi(\xi)d\xi$$

$$\Rightarrow \phi''(x) = \lambda\left[\int_0^l \frac{\partial}{\partial x}\frac{(l-\xi)}{l}\phi(\xi)d\xi\right] - \int_0^x \frac{\partial}{\partial x}[\lambda\phi(\xi)]d\xi - \lambda\phi(x)$$

$$\Rightarrow \phi''(x) + \lambda\phi(x) = 0$$

From the relation (6), we have

$\phi(0) = 0 \; and \; \phi(1) = \int_0^l \left\{\frac{\lambda l(l-\xi)}{l} - \lambda(l-\xi)\right\} \phi(\xi) d\xi = 0$ … (8)

Which is the required given differential equation.

Example: Obtain Fredholm integral equation of second kind corresponding to the boundary **value problems.**

$$\frac{d^2\phi}{dx^2} = x - \lambda\phi; \phi(0) = 0, \phi'(1) = 0$$

Also, recover the boundary value problem from the integral equation you obtain.

Solution: The differential equation may be written as

$$\frac{d^2\phi}{dx^2} = x - \lambda\phi$$

Integrating, both the sides, with regard to x, we have

$$\int_0^x \frac{d^2\phi}{dx^2} dx = \int_0^x x dx - \lambda \int_0^x \phi(x) dx$$

$$\Rightarrow \phi'(x) - \phi'(0) = \frac{1}{2}x^2 - \lambda \int_0^x \phi(\xi) d\xi$$

Integrating both the sides with regard to x, we have

$$\phi(x) - \phi(0) = Cx + \left(\frac{1}{6}\right)x^3 - \lambda \int_0^x \phi(\xi) d\xi^2$$

$$\Rightarrow \phi(x) = Cx + \left(\frac{1}{6}\right)x^3 - \lambda \int_0^x \phi(\xi) d\xi^2$$

$$\Rightarrow \phi(x) = Cx + \left(\frac{1}{6}\right)x^3 - \lambda \int_0^x (x-\xi)\, \phi(\xi) d\xi$$

Since $\phi'(1) = 0 \Rightarrow \phi'(1) = C + \frac{1}{2} - \lambda \int_0^1 \phi(\xi) d\xi$

$$\Rightarrow C = -\frac{1}{2} + \lambda \int_0^1 \phi(\xi) d\xi$$

$$\Rightarrow \phi(x) = -\left(\frac{1}{2}\right)x + \left(\frac{1}{6}\right)x^3 + \lambda \int_0^x \phi(\xi) d\xi - \lambda \int_0^x (x-\xi)\, \phi(\xi) d\xi$$

$$\Rightarrow \phi(x) = -\left(\frac{1}{2}\right)x + \left(\frac{1}{6}\right)x^3 + \lambda \left\{\int_0^x x\phi(x) d\xi + \int_0^x x\phi(\xi) d\xi\right\} - \lambda \int_0^x (x-\xi)\, \phi(\xi) d\xi$$

$$\Rightarrow \phi(x) = \left(\frac{1}{6}\right)(x^3 - 3x) + \lambda \left\{\int_0^x \xi\phi(\xi) d\xi + \int_0^x x\phi(\xi) d\xi\right\}$$

$\phi(x) = \left(\frac{1}{6}\right)(x^3 - 3x) + \lambda \int_0^x K\,(x,\xi)\phi(\xi) d\xi$ … (1)

Where $K(x,\xi) = \{x, x < \xi \;\; \xi, x > \xi$

This determined the non-homogeneous Fredholm integral equation of second. Put $I = 1$ everywhere.

Converse: The integral equation may be taken as

$\phi(x) = -\left(\frac{1}{2}\right)x + \left(\frac{1}{6}\right)x^3 + \lambda \int_0^x x\phi\,(\xi) d\xi - \lambda \int_0^x (x-\xi)\phi\,(\xi) d\xi$ … (2)

Differentiating (2) with regard to x, both the sides, we have

$$\phi'(x) = -\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)x^2 + \lambda\frac{d}{dx}\int_0^x x\phi(\xi)d\xi - \lambda\frac{d}{dx}\int_0^x (x-\xi)\phi(\xi)d\xi$$

$$\Rightarrow \phi'(x) = -\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)x^2 + \lambda\int_0^x \phi(\xi)d\xi - \lambda\int_0^x \phi(\xi)d\xi \quad \ldots (3)$$

Differentiating again with regard to x, both the sides, we have

$$\phi''(x) = x + \lambda\frac{d}{dx}\int_0^x \phi(\xi)d\xi - \lambda\frac{d}{dx}\int_0^x \phi(\xi)d\xi$$

$$\Rightarrow \phi''(x) = x + \lambda\int_0^x \frac{\partial}{\partial x}\{\phi(\xi)\}d\xi - \lambda\int_0^x \frac{\partial}{\partial x}\{\phi(\xi)\}d\xi - \lambda\phi(x)$$

$$\Rightarrow \phi''(x) = x - \lambda\phi(x)$$

$$\Rightarrow \phi''(x) + \lambda\phi(x) = x \quad \ldots (4)$$

From the relations (2) and (3), we have

$$\phi(0) = 0 \text{ and } \phi'(1) = -\frac{1}{2} + \frac{1}{2} = 0 \quad \ldots (5)$$

Volterra's Integral Equation

Definition

Solution of non-homogeneous Volterra's integral equation of second kind by the method of successive substitution.

Consider the Volterra's integral equation of second kind as

$$\phi(x) = F(x) + \lambda\int_a^x K(x,\xi)\phi(\xi)d\xi, \quad \ldots (1)$$

Where

(i) The kernel $K(x,\xi) \neq 0$ is real and continuous in the rectangle $R: a \le x \le b, a \le \xi \le b$. Consider $|K(x,\xi)| \le P$, where P is the maximum value in R.

(ii) The function $F(x) \neq 0$ is real and continuous in an interval $a \le x \le b$. Consider $|F(x)| \le Q$, where Q is the maximum value in the interval.

(iii) λ is a non-zero numerical parameter.

Substituting the unknown function $\phi(\xi)$ under an integral sign from the equation (1) itself, we have

$$\phi(x) = F(x) + \lambda\int_a^x K(x,\xi)\left[F(\xi) + \lambda\int_a^\xi K(\xi,\xi_1)\phi(\xi_1)d\xi_1\right]d\xi$$

$$\phi(x) = F(x) + \lambda\int_a^x K(x,\xi)F(\xi)d\xi + \lambda^2\int_a^x K(x,\xi)\int_a^\xi K(\xi,\xi_1)\phi(\xi_1)d\xi_1 d\xi$$

Performing the operation successively for $\phi(\xi)$, we have

$$\phi(x) = F(x) + \lambda\int_a^x K(x,\xi)F(\xi)d\xi + \lambda^2$$

$$\int_a^x K(x,\xi)\int_a^\xi K(\xi,\xi_1)\left[F(\xi_1) + \lambda\int_a^{\xi_1} K(\xi_1,\xi_2)\phi(\xi_2)d\xi_2\right]d\xi_1 d\xi$$

$$\Rightarrow \phi(x) = F(x) + \lambda \int_a^x K(x,\xi)\, F(\xi) d\xi + \lambda^2 \int_a^x K(x,\xi) \int_a^{\xi} K(\xi,\xi_1)\, F(\xi_1) d\xi_1 d\xi$$

$$+\lambda^3 \int_a^x K\,(x,\xi) \int_a^{\xi} K\,(\xi,\xi_1) \int_a^{\xi_1} K\,(\xi_1,\xi_2) \phi(\xi_2) d\xi_2 d\xi_1 d\xi + \dots\dots$$

In general, we have

$$\phi(x) = F(x) + \lambda \int_a^x K\,(x,\xi) F(\xi) d\xi + \lambda^2 \int_a^x K\,(x,\xi)$$

$$\int_a^{\xi} K\,(\xi,\xi_1) F(\xi_1) d\xi_1 d\xi + \lambda^3 \int_a^x K\,(x,\xi)$$

$$\phi(x) = F(x) + \lambda \int_a K\,(x,\xi) F(\xi) d\xi +$$

$$\lambda^2 \int_a^b K\,(x,\xi) \int_a^b K\,(\xi,\xi_1) \left\{ F(\xi_1) + \lambda \int_a^b K\,(\xi_1,\xi_2) \phi(\xi_2) d\xi_2 \right\} d\xi_1 d\xi +$$

$$\phi(x) = F(x) + \lambda \int_a^b K\,(x,\xi) F(\xi) d\xi + \lambda^2 \int_a^b K\,(x,\xi) \int_a^b K\,(\xi,\xi_1) F(\xi_1) d\xi_1 d\xi +$$

$$\lambda^3 \int_a^b K\,(x,\xi) \int_a^b K\,(\xi,\xi_1) \int_a^b K\,(\xi_1,\xi_2) \phi(\xi_2) d\xi_2 d\xi_1 d\xi + \dots +$$

$$\lambda^n \int_a^b K\,(x,\xi) \int_a^b K\,(\xi,\xi_1) \int_a^b K\,(\xi_1,\xi_2) \dots \int_a^b K\,(\xi_{n-2},\xi_{n-1}) F(\xi_{n-1}) d\xi_{n-1} \dots d\xi_1 d\xi$$

$$+\lambda^{n+1} \int_a^b K\,(x,\xi) \int_a^b K\,(\xi,\xi_1) \dots \int_a^b K\,(\xi_{n-1},\xi_n) F(\xi_n) d\xi_n \dots d\xi_1 d\xi \qquad \dots (2)$$

Consider the infinite series

$$\phi(x) = F(x) + \lambda \int_a^b K\,(x,\xi) F(\xi) d\xi + \lambda^2 \int_a^b K\,(x,\xi)$$

$$\int_a^b K\,(\xi,\xi_1) F(\xi_1) d\xi_1 d\xi + \cdots \qquad \dots (3)$$

As the kernel $K(x,\xi)$ and the known function $F(\xi)$ are real and continuous, so each term of the Above series represents a continuous, function in I, provided it converges uniformly in that interval.
Since $|K(x,\xi)| \le P$ and $|F(x)| \le Q$

Contains the maximum value in R and I respectively.
Assume $S_n(x) = \lambda^n \int_a^b K\,(x,\xi)$

$$\int_a^b K\,(\xi,\xi_1) \dots\dots \int_a^b K\,(\xi_{n-2},\xi_{n-1}) F(\xi_{n-1}) d\xi_{n-1} \dots . d\xi_1 d\xi$$

Then $|S_n(x)| \le |\lambda|^n |Q| P^n (b-a)^n$
It will converge only if

$$|\lambda| P(b-a) < 1 \Rightarrow |\lambda| < \frac{1}{P(b-a)}$$

Thus the series (2) converges absolutely and uniformly when the relation (3) holds
Again, let $S_{n+1}(x) = \lambda^{n+1} \int_a^b K(x,\xi)$

$$\int_a^b K\,(\xi,\xi_1) \dots\dots \int_a^b K\,(\xi_{n-1},\xi_n) F(\xi_n) d\xi_n$$

or $|S_{n+1}(x)| < |\lambda^{n+1}| M P^{n+1} (b-a)^{n+1}$

Solution of non-homogeneous Volterra's integra equation of second kind.

A volterra integral equation of second kind

$\phi(x) = F(x) + \lambda \int_0^x K(x,\xi)\phi(\xi)d\xi$ has one and only one solution, given by the relation

$$\phi(x) = F(x) + \lambda \int_0^x R(x,\xi;\lambda)F(\xi)d\xi$$

Where the resolvent kernel $R(x,\xi;\lambda)$ is the sum of the series $R(x,\xi;\lambda) = K(x,\xi) + \sum_{n=1}^{\infty} \lambda^n K_n(x,\xi)$, convergent for all values of λ.

Consider the Volterra integral equation of second kind

$$\phi(x) = F(x) + \lambda \int_0^x K(x,\xi)\phi(\xi)d\xi \qquad \ldots (1)$$

Where the kernel $K(x,\xi)$ is a continuous function for $0 \le x \le a, 0 \le \xi \le a$, and the function $F(x)$ is continuous for $0 \le x \le a$.

Consider an infinite power series in ascending powers of λ as

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots . + \lambda^n\phi_n(x) + \cdots \qquad \ldots (2)$$

Let the series (2) is a solution of the integral equation (1), then

$$\phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots . + \lambda^n\phi_n(x) + \cdots .$$

$$= F(x) + \lambda \int_0^x K(x,\xi)[\phi_0(\xi) + \lambda\phi_1(\xi) + \lambda^2\phi_2(\xi) + \cdots . + \lambda^n\phi_n(\xi)]d\xi \qquad \ldots (3)$$

Equating the coefficients of like power of λ, we get

$$\phi_0(x) = F(x)$$

$$\phi_1(x) = \int_0^x K(x,\xi)\phi_0(\xi)d\xi$$

$$\phi_2(x) = \int_0^x K(x,\xi)\phi_1(\xi)d\xi$$

$$\phi_n(x) = \int_0^x K(x,\xi)\phi_{n-1}(\xi)d\xi \qquad \ldots (4)$$

Thus it yields a method for a successive approximation of the function $\phi(x)$. It may be shown that the series (2) converges uniformly in x and λ, for any λ and $x \in [0,a]$, under these assumptions with regard to $F(x)$ and $K(x,\xi)$, its sum is a unique solution of the equation (1).

Further, from (4), it follows that

$\phi_1(x) = \int_0^x K(x,\xi)F(\xi)d\xi$

$\phi_2(x) = \int_0^x K(x,\xi)\left\{\int_0^\xi K(\xi,\xi_1)F(\xi_1)d\xi_1\right\}d\xi$

Here $\xi_1 = 0, \xi_1 = \xi; \xi = 0, \xi = x$

By interchanging the order of integration, we have

$$\phi_2(x) = \int_0^x F(\xi_1)d\xi_1\left\{\int_{\xi_1}^x K(x,\xi)K(\xi,\xi_1)d\xi\right\}$$

$\Rightarrow \phi_2(x) = \int_0^x K_2(x,\xi_1)F(\xi_1)d\xi_1 \qquad \ldots (5)$

Where, $K_2(x,\xi_1) = \int_{\xi_1}^x K(x,\xi)K(\xi,\xi_1)d\xi \qquad \ldots (6)$

In general, we have

$$\phi_n(x) = \int_0^x K_n(x,\xi)F(\xi)d\xi, n = 1,2, \ldots \ldots \qquad \ldots (7)$$

The functions $K_n(x,\xi)$ are called iterated kernels, which can readily be shown that

$$K_1(x,\xi) = K(x,\xi)$$

and $K_2(x,\xi) = K_3(x,\xi)$ etc., are defined recursively by the-formulas

$K_{n+1}(x,\xi) = \int_{\xi}^{x} K\,(x,z)K_n(z,\xi)dz, n = 1,2,3, \dots \dots$ … (8)

The relation (2), which represents the solution of the integral equation, (1) can therefore be written as

$\phi(x) = F(x) + \sum_{v=1}^{\infty} \lambda^{v-1} \int_0^x K_v\,(x,\xi)F(\xi)d\xi$ … (9)

Where

$R(x,\xi,\lambda) = K_1(x,\xi) + \lambda K_2(x,\xi) + \lambda^2 K_3(x,\xi) + \cdots + \lambda^{n+1}K_n(x,\xi) + \lambda^n K_{n+1}(x,\xi) + \cdots$ … (10)

$\Rightarrow R(x,\xi,\lambda) = \sum_{v=0}^{\infty} \lambda^v \int_0^x K_{v+1}\,(x,\xi),$

The function $R(x,\xi,\lambda)$ is called the resolvent kernel or reciprocal kernel of the integral equation (1). The series converges absolutely and uniformly in the case of a continuous kernel $K(x,\xi)$.
Iterated kernels and the resolvent kernel do not depend on the lower limit of an integral equation. The resolvent kernel $R(x,\xi,\lambda)$ satisfies the functional equation.

$R(x,\xi,\lambda) = K(x,\xi) + \lambda \int_{\xi}^{x} K\,(x,z)R(z,\xi,\lambda)dz.$

Thus the solution of the equation (1) reduces to

$$\phi(x) = F(x) + \lambda \int_0^x R(x,\xi,\lambda)F(\xi)d\xi$$

Example: 1 With the aid of the resolvent kernel find the solutior of the integral equation

(a) $\boldsymbol{\phi(x) = x + \int_0^x (\xi - x)\phi(\xi)d\xi}$

Solution: (i) Here $K_1(x,\xi) = K(x,\xi) = \xi - x$ … (1)

and $K_v(x,\xi) = \int_{\xi}^{x} K(x,z)K_{v-1}(z,\xi)dz$ … (2)

Substituting $v = 2,3,4,$ ..... in the relation (2), we have

$K_2(x,\xi) = \int_{\xi}^{x} K(x,z)K_1(z,\xi)dz$

$\Rightarrow K_2(x,\xi) = \int_{\xi}^{x} (z-x)(\xi - z)dz = -\frac{1}{3!}(\xi - x)^3$

or $K_3(x,\xi) = \int_{\xi}^{x} K(x,z)K_2(z,\xi)dz$

$\Rightarrow K_3(x,\xi) = \int_{\xi}^{x} (z-x)\left\{-\frac{1}{3!}(\xi - z)^3\right\}dz = -\frac{1}{5!}(\xi - x)^5$

By mathematical induction, we have

$$K_v(x,\xi) = (-1)^{v-1}\frac{1}{(2v-1)!}(\xi - x)^{2v-1}, v = 1,2,3, \dots$$

The resolvent kernel is defined as

$$R(x,\xi,\lambda) = \sum_{v=1}^{\infty} \lambda^{v-1}K_v(x,\xi) = \sum_{v=1}^{\infty} K_v(x,\xi)\lambda^{v-1}$$

$$\Rightarrow R(x,\xi,\lambda) = \frac{(\xi - x)}{1!} - \frac{(\xi - x)^3}{3!} + \frac{(\xi - x)^5}{5!} - \cdots$$

$$\Rightarrow R(x,\xi,\lambda) = sin\,(\xi - x)$$

The solution of the integral equation is determined as

$$\phi(x) = f(x) + \lambda\int_0^x R(x,\xi,\lambda) f(\xi) d\xi$$

$$\Rightarrow \phi(x) = x + \int_0^x \xi \sin(\xi - x) d\xi$$

Example: 2 Solve the integral equation $\phi(x) = 1 + \int_0^x \phi(\xi) d\xi$

Solution: Assume $\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots . + \lambda^n\phi_n(x) + \cdots$.

Here $f(x) = 1, K(x,\xi) = 1, \lambda = 1$

Equating the coefficients of like powers of λ, we have

$$\phi_0(x) = 1$$

$$\phi_1(x) = \int_0^x d\xi = x$$

$$\phi_2(x) = \int_0^x \xi d\xi = \frac{1}{2}x^2$$

$$\phi_3(x) = \frac{1}{2}\int_0^x \xi^2 d\xi = \frac{1}{3!}x^3$$

and so on.

Thus the solution of the integral equation is given by

$$\phi(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = e^x$$

Determination of Resolvent Kernels

(a) Consider that the kernel $K(x,\xi)$ is a polynomial of degree $(n-1)$ in ξ such that it may be expressed in the form.

$$K(x,\xi) = a_0(x) + a_1(x)(x-\xi) + a_2(x)\frac{1}{2!}(x-\xi)^2 + \cdots + \frac{a_{n-1}(x)}{(n-1)!}(x-\xi)^{n-1} + \cdots \ldots \qquad \ldots (1)$$

Where the coefficients $\sum_{i=0}^{n-1} a_i(x)$ are continuous in the integral $[0,a]$.

Let the auxiliary function be

$$\phi(x,\xi,\lambda) = \frac{1}{(n-1)!}(x-\xi)^{n-1} + \lambda\int_\xi^x R(t,\xi,\lambda)\frac{(x-t)^{n-1}}{(n-1)!}dt \qquad \ldots (2)$$

with the conditions

$$\phi = \frac{d\phi}{dx} = \cdots = \frac{d^{n-2}\phi}{dx^{n-2}} = 0 \; at \; x = \xi \; and \; \frac{d^{n-1}\phi}{dx^{n-1}} = 1 \; at \; x = \xi \qquad \ldots (3)$$

In addition, we have

$$R(x,\xi,\lambda) = \frac{1}{\lambda}\frac{d^n}{dx^n}\phi(x,\xi,\lambda) \qquad \ldots (4)$$

Since the resolvent kernel satisfies the functional equation

$$R(x,\xi,\lambda) = K(x,\xi) + \lambda\int_\xi^x K(x,z)R(z,\xi,\lambda)dz \qquad \ldots (5)$$

From (4) and (5), we have

$$\frac{d^n}{dx^n}\phi(x,\xi,\lambda) = \lambda K(x,\xi) + \lambda\int_\xi^x K(x,z)\frac{d^n}{dz^n}\phi(z,\xi,\lambda)dz \qquad \ldots (6)$$

$$\frac{d^n}{dx^n}\phi(x,\xi,\lambda) = \lambda K(x,\xi) + \lambda\left[K(x,z)\frac{d^{n-1}\phi}{dz^{n-1}} - \frac{\partial K(x,z)}{\partial z}\frac{d^{n-2}\phi}{dz^{n-2}} + \cdots + (-1)^{n-1}\frac{d^{n-1}K}{dz^{n-1}}\phi\right]_{z=\xi} \quad \ldots (7)$$

Using (1) and (3) the relation (7) reduces to

$$D\phi = \frac{d^n\phi}{dx^n} - \lambda\left[a_0(x)\frac{d^{n-1}\phi}{dx^{n-1}} + a_1(x)\frac{d^{n-2}\phi}{dx^{n-2}} + \cdots + a_{n-1}(x)\phi\right] = 0 \quad \ldots (8)$$

The function $\phi(x,\xi,\lambda)$ is therefore the integral of the linear equation $D\phi = 0$ which satisfies the Cauchy conditions.

Thus, we have an expression for the resolvent kernel as

$$R(x,\xi,\lambda) = \frac{1}{\lambda}\frac{d^n}{dx^n}\phi(x,\xi,\lambda) \quad \ldots (9)$$

(b) Further, assume that the kernel $K(x,\xi)$ is a polynonial of degret $(n-1)$ in x such that it may be expressed in the form.

$$K(x,\xi) = b_0(\xi) + b_1(\xi)(\xi - x) + b_2(\xi)\frac{1}{2!}(\xi - x)^2 + \cdots. + \frac{b_{n-1}(\xi)}{(n-1)!}(\xi - x)^{n-1} + \cdots \quad \ldots (10)$$

Where the coefficients $b_v(\xi)$ are continuous in the interval $[0,a]$

Consider $R(x,\xi;\lambda) = -\frac{1}{\lambda}\frac{d^n}{dx^n}\phi(x,\xi;\lambda)$ ... (11)

The auxiliary function $\phi(x,\xi;\lambda)$ satisfies the following conditions.

$$\phi = \frac{d\phi}{d\xi} = \cdots .. = \frac{d^{n-2}\phi}{d\xi^{n-2}} = 0 \text{ at } \xi = x; \frac{d^{n-1}\phi}{d\xi^{n-1}} = 1 \text{ at } \xi \neq x. \quad \ldots (12)$$

Therefore the functional relation reduces to

$$\frac{d^n\phi}{d\xi^n} = -\lambda K(x,\xi) + \lambda\int_\xi^x K(z,\xi)\frac{d^n}{d\xi^n}\phi(x,z;\lambda)dz \quad \ldots (13)$$

Using the expression (1) and (12) and integrating by parts to the integral on R.H.S., we have

$$D\phi = \frac{d^n\phi}{d\xi^n} + \lambda\left[b_0(\xi)\frac{d^{n-1}\phi}{d\xi^{n-1}} + b_1(\xi)\frac{d^{n-2}\phi}{d\xi^{n-2}} + \cdots. + b_{n-1}(\xi)\phi\right] = 0 \quad \ldots (14)$$

Thus the auxiliary function $\phi(x,\xi;\lambda)$ is the integral of the linear equation $D\phi = 0$ which satisfies the Cauchy conditions. Hence the resolvent kernel is of the form

$$R(x,\xi;\lambda) = -\frac{1}{\lambda}\frac{d^n}{d\xi^n}\phi(\xi,x;\lambda) \quad \ldots (15)$$

Example: Find the resolvent kernels of integral equations with the following kernel: ($\lambda = 1$)

$$K(x,\xi) = -2 + 3(x - \xi); \lambda = 1$$

Solution: Here $K(x,\xi) = -2 + 3(x - \xi); \lambda = 1$

Comparing with the relation, we have $a_0(x) = -2, a_1(x) = 3$; and all the other $a_v(x) = 0$

The differential equation (8) reduces to

$$\frac{d^2\phi}{dx^2} + 2\frac{d\phi}{dx} - 3\phi = 0 \quad \ldots (1)$$

with the conditions

$$\phi = 0 \; at \; x = \xi, \frac{d\phi}{dx} = 1 \; at \; x = \xi \quad \ldots (2)$$

The solution of equation (1) is given by

$$\phi = A(\xi)e^{-3x} + B(\xi)e^x \quad \ldots (3)$$

From (2) and (3), we obtain

Which is the required differential equation together with the boundary conditions.

EXERCISE : 1

1. **Form an integral equation corresponding to the differential equation.**
$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ with the initial conditions
$y(0) = 0, y'(0) = -1$

2. **From the integral equations corresponding to the following differential equations with given initial conditions.**

(a) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0; y(0) = c_1, y'(0) = c_2, y''(0) = c_3$

(b) $\frac{d^2y}{dx^2} + (1 + x^2)y = cos\ x; y(0) = 0, y'(0) = 2$

(c) $\phi''' - sin\ x\phi' + e^x\phi = x, \phi(0) = 1, \phi'(0) = -1$

(d) $\frac{d^2\phi}{dx^2} + \lambda\phi = 0; \phi(0) = 0, \phi(l) = 0$

$\phi(x, \xi; 1) = \frac{1}{4}e^{x-\xi} - \frac{9}{4}e^{-3(x-\xi)}$

Hence the resolvent kernel is given by

$R(x, \xi; 1) = \frac{1}{\lambda}\frac{d^2}{dx^2}\phi(x, \xi; 1) = \frac{1}{4}e^{x-\xi} - \frac{1}{4}e^{-3(x-\xi)}$

Example: Solve the Volterra's integral equation

$\phi(x) = (1 - 2x - 4x^2) + \int_0^x [3 + 6(x - \xi) - 4(x - \xi)^2]\phi(\xi)d\xi$

Solution: Here $f(x) = 1 - 2x - 4x^2; \lambda = 1;$

$$K(x, \xi) = 3 + 6(x - \xi) - 4(x - \xi)^2$$

Comparing with the relation (1), we have

$$a_0(x) = 3, a_1(x) = 6, a_2(x) = -8$$

The $D\phi = \frac{d^n\phi}{dx^n} - \lambda\left[a_0(x)\frac{d^{n-1}\phi}{dx^{n-1}} + a_1(x)\frac{d^{n-2}\phi}{dx^{n-2}} + \cdots + a_{n-1}(x)\phi\right] = 0$ reduces to

$\frac{d^3\phi}{dx^3} - 3\frac{d^2\phi}{dx^2} - 6\frac{d\phi}{dx} + 8\phi = 0$ ... (1)

with the conditions

$\phi = 0 = \frac{d\phi}{dx}$ at $x = \xi$ and $\frac{d^2\phi}{dx^2} = 1$ at $x = \xi$ ... (2)

The solution of equation (1) is given by

$\phi(x, \xi; 1) = A(\xi)e^x + B(\xi)e^{-2x} + C(\xi)e^{4x}$ ... (3)

From (2) and (3), we obtain

$$R(x, \xi; 1) = \frac{1}{\lambda}\frac{d^3}{dx^3}\phi(x, \xi; 1)$$

Therefore, the resolvent becomes

$$R(x, \xi; 1) = \frac{1}{\lambda}\frac{d^3}{dx^3}\phi(x, \xi; 1)$$

$= -\frac{1}{9}\left[e^{x-\xi} + 4e^{-2(x-\xi)} - 32e^{4(x-\xi)}\right]$ Thus, the solution of the integral equation is given by

$$\phi(x) = 1 - 2x - 4x^2 - \frac{1}{9}\int_0^x e^{x-\xi}(9 + 10\xi + 4\xi^2) + e^{x-\xi}(4\xi - 8\xi^2) + e^{x-\xi}(-32\xi - 32\xi^2)d\xi$$

or $\phi(x) = 1 - 2x - 4x^2 - 1 + 2x + 4x^2 + e^x = e^x$

Example: Solve the Volterra integral equation of second kind, by using the method of successive approximation.

$\phi(x) = (1 + x) - \int_0^x \phi(\xi)d\xi$, with $\phi_0(x) = 1$

Solution: The integral equation is given as $\phi(x) = (1 + x) - \int_0^x \phi(\xi)d$

Here $f(x) = 1 + x, K(x,\xi) = 1$ and $\lambda = -1$

The vth order approximation is given by

$$\phi_v(x) = f(x) + \lambda\int_0^x K(x,\xi)\phi_{v-1}(\xi)d\xi$$

or $\phi_v(x) = (1 + x) - \int_0^x \phi_{v-1}(\xi)d\xi$,

Substituting $v = 1,2,3,$ .... We have

$$\phi_1(x) = (1 + x) - \int_0^x \phi_0(\xi)d\xi = (1 + x) - \int_0^x d\xi = 1$$

$$\phi_2(x) = (1 + x) - \int_0^x \phi_1(\xi)d\xi = (1 + x) - x = 1$$

$$\phi_3(x) = (1 + x) - \int_0^x \phi_2(\xi)d\xi = (1 + x) - x = 1$$

$$\phi_v(x) = (1 + x) - \int_0^x \phi_{v-1}(\xi)d\xi = 1$$

Hence the solution of the integral equation is given by

$$\phi(x) = lim_{v\to\infty}\phi_v(x) = 1$$

4. Solution of the Fredholm integral equation by the method of successive substitutions.

Consider the Fredholm integral equation of second kind as

$\phi(x) = F(x) + \lambda\int_a^b K(x,\xi)\phi(\xi)d\xi$ … (1)

where

(i) The kernel $K(x,\xi) \neq 0$ is real and continuous in the rectangle $R: a \leq \xi \leq b$. Consider $|K(x,\xi)| \leq P$, where P is the maximum value in R.

(ii) The function $F(x) \neq 0$ is real and continuous in an interval $I: a \leq x \leq b$. Consider $|F(x)| \leq Q$, when Q the maximum value is in the interval.

(ii) λ is a non-zero numerical parameter.

Since there exists a continuous solution $\phi(x)$ so substituting the unknown function under an integral sign from the equation (1) itself, we obtain.

$$\phi(x) = F(x) + \lambda\int_a^b K(x,\xi)\left\{F(\xi) + \lambda\int_a^b K(\xi,\xi_1)\phi(\xi_1)d\xi_1\right\}d\xi$$

$$\phi(x) = F(x) + \lambda\int_a^b K(x,\xi)F(\xi)d\xi + \lambda^2\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\phi(\xi_1)d\xi_1 d\xi$$

Proceeding in this manner successively for $\phi(\xi)$, we get

$$\phi(x) = F(x) + \lambda \int_a^x K\,(x, \xi)F(\xi)d\xi +$$

$$\lambda^2 \int_a^x K\,(x, \xi) \int_a^{\xi_1} K\,(\xi, \xi_1)F(\xi_1)d\xi_1 d\xi + \lambda^3 \int_a^x K\,(x, \xi) \int_a^{\xi_1} K\,(\xi, \xi_1)\phi(\xi_1)d\xi_1 d\xi d\xi_1$$

$$\phi(x) = F(x) + \lambda \int_a^x K\,(x, \xi)F(\xi)d\xi + \lambda^2 \int_a^x K\,(x, \xi) \int_a^{\xi_1} K\,(\xi, \xi_1)F(\xi_1)d\xi_1 d\xi +$$

$$\lambda^3 \int_a^x K\,(x, \xi) \int_a^{\xi_1} K\,(\xi, \xi_1) \int_a^{\xi_2} K\,(\xi_1, \xi_2)\phi(\xi_2)d\xi_2 d\xi_2 d\xi + \cdots +$$

$$\lambda^n \int_a^x K\,(x, \xi) \int_a^{\xi_1} K\,(\xi, \xi_1) \int_a^{\xi_2} K\,(\xi_1, \xi_2) \ldots \int_a^{\xi_{n-1}} K\,(\xi_{n-2}, \xi_{n-1})F(\xi_{n-1})d\xi_{n-1} \ldots d\xi_1 d\xi$$

$$+\lambda^{n+1} \int_a^x K\,(x, \xi) \int_a^{\xi_1} K\,(\xi, \xi_1) \ldots \int_a^{\xi_n} K\,(\xi_{n-1}, \xi_n)F(\xi_n)d\xi_n \ldots d\xi_1 d\xi \qquad \ldots (2)$$

Consider the infinite series

$$\phi(x) = F(x) + \lambda \int_a^x K\,(x, \xi)F(\xi)d\xi \;\; + \lambda^2 \int_a^x K\,(x, \xi) \;\; \int_a^{\zeta_1} K\,(\xi, \xi_1)F(\xi_1)d\xi_1 d\xi + \cdots \qquad \ldots (3)$$

As the kernel $K(x, \xi)$ and the known function $F(\xi)$ are real and continuous, so each term of the Above series represents a continuous function in I, provided it converges uniformly in that interval.
Since $|K(x, \xi)| \leq P$ and $|F(x)| \leq Q$
Contains the maximum value in R and I respectively.
Assume $S_n(x) = \lambda^n \int_a^x K\,(x, \xi)$

$$\int_a^{\xi_1} K\,(\xi, \xi_1) \ldots \ldots \int_a^{\xi_n} K\,(\xi_{n-1}, \xi_n)F(\xi_n)d\xi_n \ldots d\xi_1 d\xi$$

Then $|S_n(x)| \leq |\lambda|^n |QP^n(b-a)^n|$
It will converge only if

$$|\lambda|P(b-a) < 1 \Rightarrow |\lambda| < \frac{1}{P(b-a)}$$

Thus the series (2) converges absolutely and uniformly when the relation (3) holds

Again, let $S_{n+1}(x) = \lambda^{n+1} \int_a^x K(x, \xi)$

$$\int_a^{\xi_1} K\,(\xi, \xi_1) \ldots \ldots \int_a^{\xi_n} K\,(\xi_{n-1}, \xi_n)F(\xi_n)d\xi_n \ldots d\xi_1 d\xi$$

or $|S_{n+1}(x)| < |\lambda^{n+1}| MP^{n+1}(b-a)^{n+1}$
Where M is the maximum value of the absolute value of the function $\phi(x)$ in I.
If $|\lambda|P(b-a) < 1$ then $\lim_{n \to \infty} S_{n+1}(x) = 0$

Thus we notice that the function $\phi(x)$ which satisfies the relation (2) is the continuous function given by the series (1). We can verify the direct substitution that the function $\phi(x)$ defined by (2) satisfies the integral equation (1). Multiplying (2) both the sides with $\lambda K(x, \xi)$ and integrating term by term within the fixed domain, we have

$$\lambda \int_a^x K\,(x, \xi)\phi(\xi)d\xi = \lambda \int_a^x K\,(x, \xi)[F(\xi) + \lambda] \int_a^x K\,(\xi, \xi_1)$$